Confluence of Conditional Rewriting Modulo

– Invited talk –

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Equational Generalized Term Rewriting Systems $\mathcal{R}$ *(EGTRSs)* consist of [Luc24, Luc25]:

- Conditional equations (E)
- Horn clauses (H)
- Conditional rewrite rules (R)

$$xs +\!\!+ (ys +\!\!+ zs) = (xs +\!\!+ ys) +\!\!+ zs$$

$$\mathsf{Nat}(0)$$

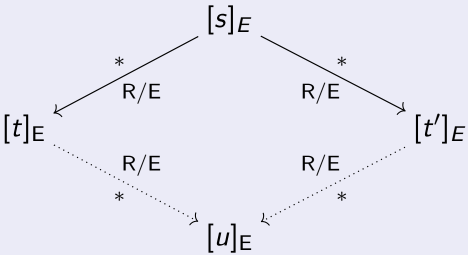
$$\mathsf{Nat}(\mathsf{s}(n)) \Leftarrow \mathsf{Nat}(n)$$

$$x \approx y \Leftarrow x \to^* y$$

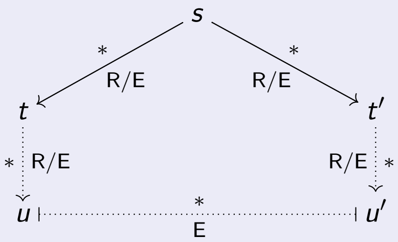
$$\begin{array}{rcl} 0 + n & \to & n \\ \mathsf{s}(m) + n & \to & \mathsf{s}(m + n) \\ \mathsf{sum}(m) & \to & n \Leftarrow m \approx n, \mathsf{Nat}(n) \\ \mathsf{sum}(ms) & \to & m + n \\ & \Leftarrow & ms \approx m +\!\!+ ns, \mathsf{Nat}(m), \\ & & \mathsf{sum}(ns) \approx n \end{array}$$

$$\mathsf{sum}(\overbrace{\underbrace{0 +\!\!+ \mathsf{s}(0) +\!\!+ \mathsf{s}(\mathsf{s}(0))}_{[0 +\!\!+ \mathsf{s}(0)] +\!\!+ \mathsf{s}(\mathsf{s}(0))}}^{0 +\!\!+ [\mathsf{s}(0) +\!\!+ \mathsf{s}(\mathsf{s}(0))]}) \to_{\mathcal{R}/E} 0 + \mathsf{s}(\mathsf{s}(\mathsf{s}(0))) \to_{\mathcal{R}/E} \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))$$

E-confluence (CR(R/E)) is confluence of rewriting on *equivalence classes*



Equivalent to the commutation of the following diagram on *terms*



- Lambda calculus (β-*reduction* + α-*conversion*) [CR36, Hin64, Hin69]
- Code optimization [ASU72]
- Rewriting-based *equational reasoning* [KB70, Set74, Hue80, Jou83]
- Theorem proving *modulo equations* [Dow99, DHK03]
- Programming languages implementing *rewriting on equivalence classes* (e.g., Maude) [Mes92, Mes12, CDE^{+}07]
- Extensions to Logically Constrained Term Rewriting Systems [ANS24], Nominal Rewriting [FNSS25], etc.
- ...

Abstract approaches

Early abstract approaches

Newman [New42], Hindley [Hin64, Hin69], Rosen [Ros70, Ros73], Aho, Sethi, Ullman [ASU72], Sethi [Set74], Huet [Hue77, Hue80], Jouannaud [Jou83]

Definitions [JK86]:

Let $\vdash\!\dashv_{\mathsf{E}}$ be a *symmetric* relation on A

Let R and R^{E} be reduction relations on A

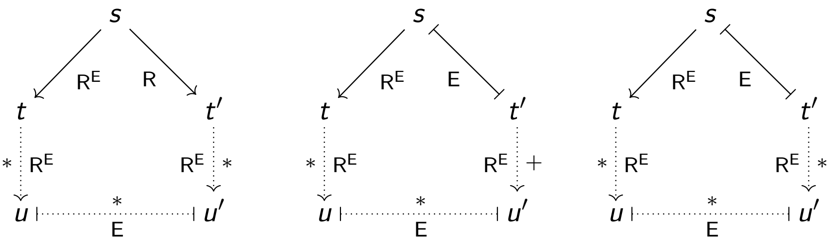
Assumptions:

J&K1: $\sim_{\mathsf{E}}$ is $\vdash\!\dashv_{\mathsf{E}}^{*}$

J&K2: $\to_{\mathsf{R}/\mathsf{E}} = \sim_{\mathsf{E}} \circ \to_{\mathsf{R}} \circ \sim_{\mathsf{E}}$

Fundamental assumption J&K3

$$\to_{\mathsf{R}} \subseteq \to_{\mathsf{R}^{\mathsf{E}}} \subseteq \to_{\mathsf{R}/\mathsf{E}}$$



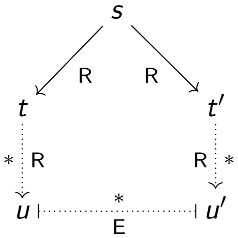
Local Confluence	*Local Coherence*	*Local Coherence*
of R^{E} modulo E with R	of R^{E} modulo E	of R^{E} modulo E
$\mathsf{LConf}_{\mathsf{E}}(\mathsf{R}^{\mathsf{E}},\mathsf{R})$	$\mathsf{LCoh}_{\mathsf{E}}(\mathsf{R}^{\mathsf{E}})$	(assume SN(R/E))

Main result cf. [JK86, Theorem 5]: An E-terminating relation R is *E-confluent* if

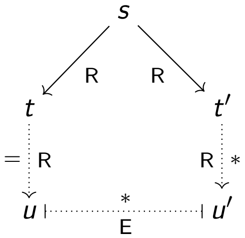
R^{E} is locally confluent with R modulo E and locally coherent modulo E.

Non-*E*-confluence: If there is a *non-* $\widetilde{\downarrow}_{\mathsf{R}/\mathsf{E}}$ *-joinable* ${}_{\mathsf{R}^{\mathsf{E}}}\Uparrow_{\mathsf{R}}$-peak, then R is *not E-confluent*. (Note: ${}_{\mathsf{R}^{\mathsf{E}}}\Uparrow_{\mathsf{E}}$-peaks are always $\widetilde{\downarrow}_{\mathsf{R}/\mathsf{E}}$ -joinable!)

$\mathsf{LConf}_E(R, R)$

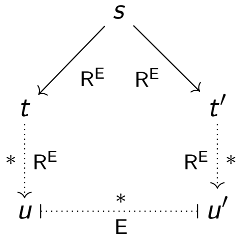


[Set74, P3]
[Hue80, Property α]
[Ohl98, LCON$\sim$]



[Ohl98, SLCON$\sim$]

$\mathsf{LConf}_E(R^E, R^E)$

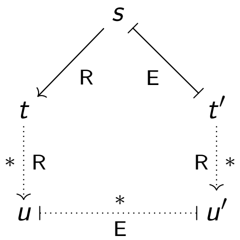


[DM12]

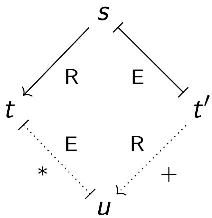
$$\mathsf{SLCON}\sim \;\Rightarrow\; \mathsf{LConf}_E(R,R) \;\Rightarrow\; \mathsf{LConf}_E(R^E,R) \;\Rightarrow\; \mathsf{LConf}_E(R^E,R^E)$$

$$\mathsf{LCoh}_E(R) \;\Rightarrow\; \mathsf{LCoh}_E(R^E)$$

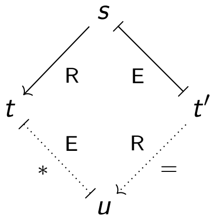
$\mathsf{LCoh_E(R)}$ & SN(R/E)



[Hue80, Property γ]
[Ohl98, LCOH⊢⊣]



[JM84, Def. 11]
[Ohl98, LCMU⊢⊣]

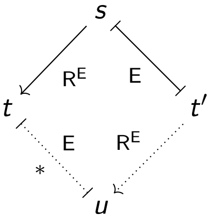


[Ohl98, SCOM⊢⊣]

SCOM⊢⊣ $\Rightarrow$ LCOH⊢⊣

LCMU⊢⊣ $\Rightarrow$ $\mathsf{LCoh_E(R)}$

SLCoh $\Rightarrow$ $\mathsf{LCoh_E(R^E)}$



[Mes17, p. 4]
SLCoh

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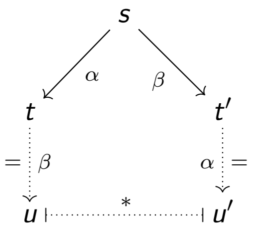
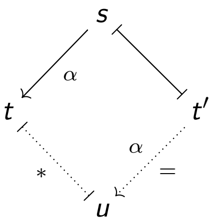
Consider *Abstract Reduction Systems* $(A, \langle \rightarrow_\alpha \rangle_{\alpha \in I}, \vdash\!\dashv)$, where I is a set of *indices*



$\rightarrow_\alpha$ subcommutes with $\rightarrow_\beta$ modulo $\sim$
[Ohl98, Fig. 8 (left)]



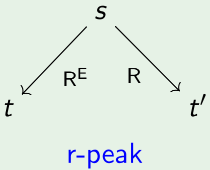
$\rightarrow_\alpha$ is SCOM$\vdash\!\dashv$
[Ohl98, Fig. 8 (right)]

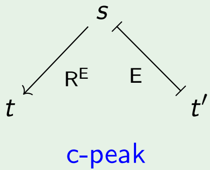
[Ohl98, Corollary 15] R is E-*confluent* if

- for all $\alpha, \beta \in I$, $\rightarrow_\alpha$ subcommutes with $\rightarrow_\beta$ modulo $\sim$; and
- $\rightarrow_\alpha$ is SCOM$\vdash\!\dashv$

[Ohl98, Corollary 20] SN(R) $\wedge$ LCON$\sim$ $\wedge$ LCMU$\vdash\!\dashv$ $\Rightarrow$ CR(R/E)

Reduction-peaks and Coherence-peaks





Application to unconditional term rewriting

- $\vdash\!\dashv_{\mathsf{E}}$ is $\to_{\overleftrightarrow{E}}$ and $\overleftrightarrow{E}$ consists of $s \to t$ and $t \to s$ for each $s = t$ in E
- $\sim_{\mathsf{E}}$ is $=_E$, i.e., $\to^*_{\overleftrightarrow{E}}$
- $s \to_{\mathcal{R},E} t$ is *Peterson & Stickel* reduction modulo [PS81]:
  - $s|_p =_E \sigma(\ell)$ for some rule $\ell \to r$ in $\mathcal{R}$ and substitution σ and
  - $t = s[\sigma(r)]_p$
- $\to_{\mathcal{R}/E}$ is $=_E \circ \to_{\mathcal{R}} \circ =_E$ (which coincides with $=_E \circ \to_{\mathcal{R},E} \circ =_E$)

Necessary conditions for E-termination of $\mathcal{R}$ [JK86, page 1169]

If $\mathcal{R}$ is E-terminating, then

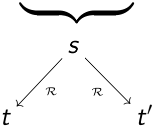
- E is *regular* [Sie89, page 243], i.e., for all $s = t \in E$, $\mathcal{V}ar(s) = \mathcal{V}ar(t)$
- E contains *no equation $x = t$, where x occurs twice in t*

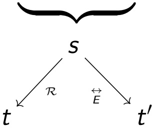
Peaks for the analysis of E-confluence of (E-terminating) $\mathcal{R}$

Parameterization	r-peak	c-peak	Joinability
R and R$^{\mathsf{E}}$ as $\to_{\mathcal{R}}$:	$t \leftarrow_{\mathcal{R}} s \to_{\mathcal{R}} t'$	$t \leftarrow_{\mathcal{R}} s \leftrightarrow_{E} t'$	$\widetilde{\downarrow}_{\mathcal{R}}$
R as $\to_{\mathcal{R}}$ and R$^{\mathsf{E}}$ as $\to_{\mathcal{R},E}$:	$t \leftarrow_{\mathcal{R},\mathsf{E}} s \to_{\mathcal{R}} t'$	$t \leftarrow_{\mathcal{R},\mathsf{E}} s \leftrightarrow_{E} t'$	$\widetilde{\downarrow}_{\mathcal{R},E}$
R and R$^{\mathsf{E}}$ as $\to_{\mathcal{R},E}$:	$t \leftarrow_{\mathcal{R},\mathsf{E}} s \to_{\mathcal{R},\mathsf{E}} t'$ can be written as $t \leftarrow_{\mathcal{R},\mathsf{E}} s' \to_{\mathcal{R}} t'$ for $s' =_E s$	$t \leftarrow_{\mathcal{R},\mathsf{E}} s \leftrightarrow_{E} t'$	$\widetilde{\downarrow}_{\mathcal{R},E}$

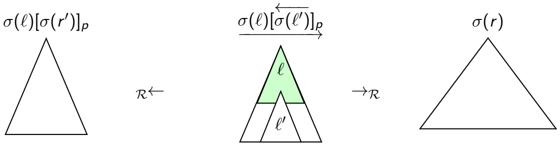
Application to unconditional term rewriting

r-peaks and c-peaks as conditional pairs



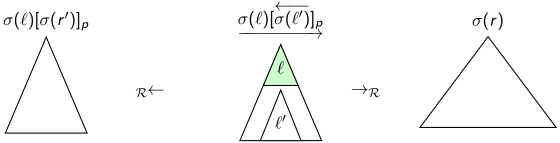


Critical r-peak
$p \in \mathcal{P}os_{\mathcal{F}}(\ell)$



$CP : \langle \theta(\ell)[\theta(r')]_p, \theta(r) \rangle$ where $\ell|_p =^?_\theta \ell'$

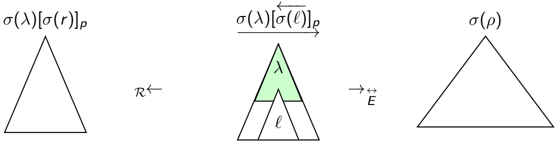
Variable r-peak
$p \notin \mathcal{P}os_{\mathcal{F}}(\ell)$



$x \in \mathcal{V}ar(\ell)$
$p \geq q \in \mathcal{P}os_x(\ell)$

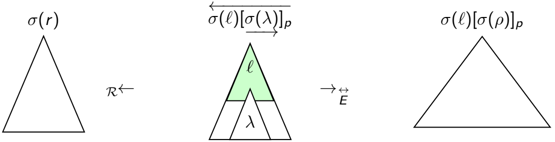
$CVP : \langle \ell[x]_q, r' \rangle \Leftarrow x \to x'$ $\widetilde{\downarrow}_{\mathcal{R}}$ *-joinable*

Critical c-up-peak
$p \in \mathcal{P}os_{\mathcal{F}}(\lambda)$



$CP : \langle \theta(\lambda)[\theta(r)]_p, \theta(\rho) \rangle$ where $\lambda|_p =^?_\theta \ell$

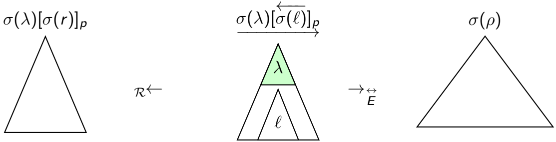
Critical c-down-peak
$p \in \mathcal{P}os_{\mathcal{F}}(\ell) - \{\Lambda\}$



$CP : \langle \theta(\ell)[\theta(\rho)]_p, \theta(r) \rangle$ where $\ell|_p =^?_\theta \lambda$

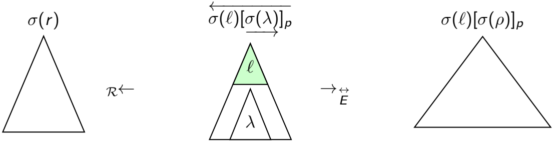
For *asymmetric* joinability criteria on *c-peaks*, e.g., $=_E \circ {}^{+}_{\mathcal{R}}\!\leftarrow$ in LCMU$\vdash$, c-up CPs join with $=_E \circ {}^{+}_{\mathcal{R}}\!\leftarrow$ and c-down CPs with $\to^{+}_{\mathcal{R}} \circ =_E$!

Variable c-up-peak
$p \notin \mathcal{P}os_{\mathcal{F}}(\lambda)$
$x \in \mathcal{V}ar(\lambda)$
$p \geq q \in \mathcal{P}os_x(\lambda)$



$CVP : \langle \lambda[x']_q, \rho \rangle \Leftarrow x \to x'$ $\widetilde{\downarrow}_{\mathcal{R}}$ *-joinable*

Variable
c-down-peak
$p \notin \mathcal{P}os_{\mathcal{F}}(\ell)$
$x \in \mathcal{V}ar(\ell)$
$p \geq q \in \mathcal{P}os_x(\ell)$



$CVP : \langle \ell[x']_q, r \rangle \Leftarrow x \dashv\vdash x'$

For *asymmetric* joinability criteria on *c-peaks*, e.g., $=_E \circ \, {}_{\mathcal{R}}^{+}\!\!\leftarrow$ in LCMU$\dashv\vdash$,
c-up CVPs join with $=_E \circ \, {}_{\mathcal{R}}^{+}\!\!\leftarrow$ and c-down CVPs with $\to_{\mathcal{R}}^{+} \circ =_E$!

Theorem (Confluence of ETRSs $\mathcal{R}$ using $\widetilde{\downarrow}_{\mathcal{R}}$ -joinability)

1. *An E-terminating ETRS $\mathcal{R}$ is E-confluent if all pairs in*

$$\underbrace{\mathsf{CP}(\mathcal{R})}_{\textit{r-peaks}} \cup \underbrace{\mathsf{CP}(E,\mathcal{R}) \cup \mathsf{CP}(\mathcal{R},E) \cup \mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})}_{\textit{c-peaks}}$$

   are $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable.

2. *An E-terminating left-linear ETRS $\mathcal{R}$ is E-confluent if all pairs in*

$$\mathsf{CP}(\mathcal{R}) \cup \mathsf{CP}(E,\mathcal{R}) \cup \mathsf{CP}(\mathcal{R},E)$$

   are $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable [Hue80, Theorem 3.3].

Theorem (Non-E-confluence of ETRSs $\mathcal{R}$)

If there is a non-$\mathcal{R}/E$-joinable $\pi \in \mathsf{CP}(\mathcal{R})$ then $\mathcal{R}$ is not E-confluent.

Example of an ETRS $\mathcal{R} = (\mathcal{F}, E, R)$ from [Luc25, Example 8.14]

Let $E = \{(1)\}$ and $R = \{(2), (3), (4), (5), (6), (7)\}$, where

$$\begin{array}{lllllllllllll} \mathsf{a} & = & \mathsf{b} & (1) & \mathsf{f}(x,x) & \to & \mathsf{g}(x) & (2) & \mathsf{b} & \to & \mathsf{c} & (5) \\ & & & & \mathsf{f}(x,x) & \to & \mathsf{h}(x) & (3) & \mathsf{g}(x) & \to & x & (6) \\ & & & & \mathsf{a} & \to & \mathsf{c} & (4) & \mathsf{h}(x) & \to & x & (7) \end{array}$$

Proof of E-termination

The associate *First-Order Theory* $\overline{\mathcal{R}}$ of $\mathcal{R}$ describes rewriting modulo:

$$s \to_{\mathcal{R}/E} t \text{ iff } \overline{\mathcal{R}} \vdash s \stackrel{rm}{\to} t$$

If there is a *model* $\mathcal{A}$ of $\overline{\mathcal{R}}$ such that $(\stackrel{rm}{\to})^{\mathcal{A}}$ is *well-founded*, then $\mathcal{R}$ is E-terminating

Such a model is obtained for $\mathcal{R}$ above with AGES [GL19]

Critical pairs

Here $\mathrm{CP}(\mathcal{R}) = \{(8)\}$, $\mathrm{CP}(\mathcal{R}, E) = \emptyset$ and $\mathrm{CP}(E, \mathcal{R}) = \{(9), (10)\}$, for

$$\pi_{(2),\Lambda,(3)} : \quad \langle \mathsf{h}(x), \mathsf{g}(x) \rangle \tag{8}$$

$$\pi_{\overrightarrow{(1)},\Lambda,(4)} : \quad \langle \mathsf{c}, \mathsf{b} \rangle \tag{9}$$

$$\pi_{\overleftarrow{(1)},\Lambda,(5)} : \quad \langle \mathsf{c}, \mathsf{a} \rangle \tag{10}$$

All these critical pairs are $\downarrow_{\mathcal{R}}$-joinable

Conditional Variable Pairs

$\mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})$ consists of six $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable CVPs. For instance,

$$\pi^{\vdash\!\dashv}_{(2),x,1} : \langle \mathsf{f}(x', x), \mathsf{g}(x)\rangle \Leftarrow x \vdash\!\dashv x'$$

if σ satisfies $x \vdash\!\dashv x'$ then

$$\sigma = \{x \mapsto C[\mathsf{a}], x' \mapsto C[\mathsf{b}]\} \qquad \text{or} \qquad \sigma = \{x \mapsto C[\mathsf{b}], x' \mapsto C[\mathsf{a}]\}$$

for some $C[\,]$ and (in the first case)

$$\begin{array}{rcc} \sigma(\mathsf{f}(x', x)) = \mathsf{f}(C[\underline{\mathsf{b}}], C[\underline{\mathsf{a}}]) \to^{+}_{\mathcal{R}} & \underline{\mathsf{f}(C[\mathsf{c}], C[\mathsf{c}])} \\ & \downarrow_{\mathcal{R}} \\ \sigma(\mathsf{g}(x)) = \mathsf{g}(C[\underline{\mathsf{a}}]) \qquad \to_{\mathcal{R}} & \mathsf{g}(C[\mathsf{c}]) \end{array}$$

and similarly for the alternative σ.

The ETRS $\mathcal{R}$ is proved *E-confluent* (Huet's result does *not* apply)

Limitations of $\widetilde{\downarrow}_{\mathcal{R}}$ -joinability with ETRSs $\mathcal{R} = (\mathcal{F}, E, R)$

Let $E = \{(11)\}$ and $R = \{(12)\}$ as in [Hue80, Remark in page 818], with

$$\mathsf{a} = \mathsf{b} \quad (11) \qquad\qquad \mathsf{f}(x, x) \rightarrow \mathsf{g}(x) \quad (12)$$

Note: $\mathsf{CP}(\mathcal{R}) = \mathsf{CP}(E, \mathcal{R}) = \mathsf{CP}(\mathcal{R}, E) = \emptyset$ and $\mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})$ consists of

$$\pi^{\vdash\!\dashv}_{(12),x,1} : \langle \mathsf{f}(x', x), \mathsf{g}(x)\rangle \Leftarrow x \vdash\!\dashv x'$$
$$\pi^{\vdash\!\dashv}_{(12),x,2} : \langle \mathsf{f}(x, x'), \mathsf{g}(x)\rangle \Leftarrow x \vdash\!\dashv x'$$

which are *not* $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable: $\sigma = \{x \mapsto \mathsf{a}, x' \mapsto \mathsf{b}\}$ satisfies $x \vdash\!\dashv x'$, but $\mathsf{f}(\mathsf{b}, \mathsf{a})$ and $\mathsf{g}(\mathsf{a})$ are *not* $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable (but they are $\widetilde{\downarrow}_{\mathcal{R},E}$ -joinable).

$\mathsf{LConf}_E(\rightarrow_{\mathcal{R}}, \rightarrow_{\mathcal{R}})$ holds but $\mathsf{LCoh}_E(\rightarrow_{\mathcal{R}})$ fails to hold

Neither E-confluence nor non-E-confluence follow from previous results.

Actually, $\mathcal{R}$ *is E-confluent*, see below

Limitations of CPs and CVPs with ETRSs $\mathcal{R} = (\mathcal{F}, E, R)$

Consider $\mathcal{R} = (\mathcal{F}, E, R)$ with $E = \{(13), (14)\}$ and $R = \{(15), (16)\}$ for

$$\begin{array}{rcll} \mathsf{b} & = & \mathsf{f}(\mathsf{a}) & (13) \\ \mathsf{a} & = & \mathsf{c} & (14) \end{array} \qquad \begin{array}{rcll} \mathsf{c} & \to & \mathsf{d} & (15) \\ \mathsf{b} & \to & \mathsf{d} & (16) \end{array}$$

from [Luc24, Example 14]. Note that $\mathsf{CP}(\mathcal{R}) = \mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R}) = \emptyset$.
$\mathsf{CP}(\mathcal{R}, E) = \emptyset$ and $\mathsf{CP}(E, \mathcal{R}) = \{(17), (18)\}$, where

$$\pi_{\overrightarrow{(13)},\Lambda,(16)} : \quad \langle \mathsf{d}, \mathsf{f}(\mathsf{a}) \rangle \quad (17)$$

$$\pi_{\overleftarrow{(14)},\Lambda,(15)} : \quad \langle \mathsf{d}, \mathsf{a} \rangle \quad (18)$$

are both $\widetilde{\downarrow}_{\mathcal{R},E}$ -joinable (but *not* $\widetilde{\downarrow}_{\mathcal{R}}$ *-joinable*): $\mathsf{f}(\mathsf{a}) =_E \mathsf{b} \to_{(16)} \mathsf{d}$ and $\mathsf{a} =_E \mathsf{c} \to_{(15)} \mathsf{d}$.

$\mathsf{LConf}_E(\to_{\mathcal{R}}, \to_{\mathcal{R}})$ holds but $\mathsf{LCoh}_E(\to_{\mathcal{R}})$ fails to hold.

Neither E-confluence nor non-E-confluence follow from previous results.

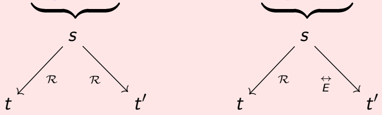
Actually, $\mathcal{R}$ is *not E-confluent*, see below

Critical and Conditional Variable Pairs of ETRSs in

$\mathsf{CP}(\mathcal{R}), \mathsf{CP}(E,\mathcal{R}), \mathsf{CP}(\mathcal{R},E)$, and $\mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})$

capturing

r-peaks and c-peaks

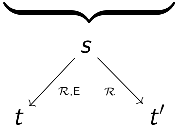


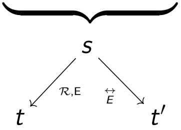
are *not enough to capture all (non-)E-confluence situations*

either by $\widetilde{\downarrow}_{\mathcal{R}}$ -joinability or by $\widetilde{\downarrow}_{\mathcal{R},E}$ -joinability

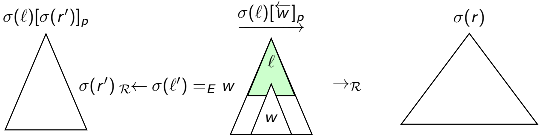
Application to unconditional term rewriting

PS-r-peaks and PS-c-peaks as conditional pairs





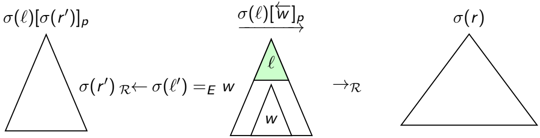
PS-r-up
Critical peak
$p \in \mathcal{P}os_{\mathcal{F}}(\ell)$



$ECP : \langle \theta(\ell)[\theta(r')]_p, \theta(r) \rangle$ — $\ell|_p =^?_{E,\theta} \ell'$ E-unifier!

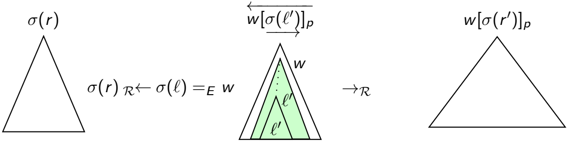
$LCCP : \langle \ell[r']_p, r \rangle \Leftarrow \ell|_p = \ell'$ — No E-unifier!

PS-r-up
Variable peak
$p \notin \mathcal{P}os_{\mathcal{F}}(\ell)$



$x \in \mathcal{V}ar(\ell)$
$p \geq q \in \mathcal{P}os_x(\ell)$

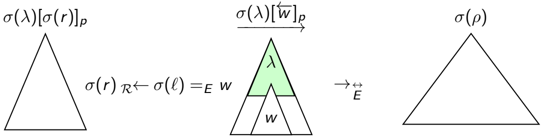
$CVP : \langle \ell[x']_q, r \rangle \Leftarrow x \overset{ps}{\to} x'$ — $\widetilde{\downarrow}_{\mathcal{R},E}$ *-joinable*



PS-r-down peak
$p > \Lambda$

$$DCP : \langle r, x' \rangle \Leftarrow x = \ell, x \xrightarrow{>\Lambda} x'$$

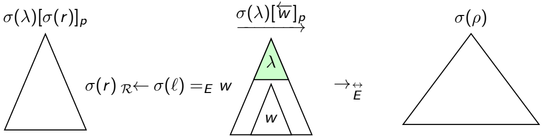
PS-c-up
Critical peak
$p \in \mathcal{P}os_{\mathcal{F}}(\lambda)$



$ECP : \langle \theta(\lambda)[\theta(r)]_p, \theta(\rho) \rangle$ $\quad \lambda|_p =^?_{E,\theta} \ell$

$LCCP : \langle \lambda[r]_p, \rho \rangle \Leftarrow \lambda|_p = \ell$

PS-c-up
Variable peak
$p \notin \mathcal{P}os_{\mathcal{F}}(\lambda)$

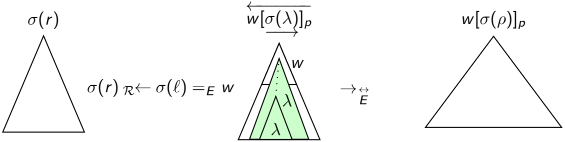


$x \in \mathcal{V}ar(\lambda)$
$p \geq q \in \mathcal{P}os_x(\lambda)$

$CVP : \langle \lambda[x']_q, \rho \rangle \Leftarrow x \overset{ps}{\to} x'$ $\quad \widetilde{\downarrow}_{\mathcal{R},E}$ *-joinable*

PS-c-down peak
$p > \Lambda$

$\downarrow_{\mathcal{R},E}$-joinable!
(no pairs needed)



Theorem (Confluence of ETRSs $\mathcal{R}$ using $\widetilde{\downarrow}_{\mathcal{R},E}$ -joinability)

An E-terminating ETRS $\mathcal{R}$ is E-confluent if all pairs in

$$\mathsf{LCCP}(\mathcal{R}) \cup \mathsf{LCCP}(E,\mathcal{R}) \qquad \textit{(resp. } \mathsf{ECP}(\mathcal{R}) \cup \mathsf{ECP}(E,\mathcal{R}))$$

are $\widetilde{\downarrow}_{\mathcal{R},E}$ -joinable (resp. $\widetilde{\downarrow}_{\mathcal{R}/E}$ -joinable [JK86, Theorem 16]).

Theorem (Non-E-confluence of ETRSs $\mathcal{R}$)

If there is a non-$\mathcal{R}/E$-joinable

$$\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{DCP}(\mathcal{R})$$

then $\mathcal{R}$ is not E-confluent

Confluence of an ETRS $\mathcal{R} = (\mathcal{F}, E, R)$ [Hue80, Remark in p. 818]

Let $E = \{(19)\}$ and $R = \{(20)\}$, with

$$\begin{array}{rcl} \mathsf{a} & = & \mathsf{b} \\ \mathsf{f}(x,x) & \to & \mathsf{g}(x) \end{array} \qquad \begin{array}{r} (19) \\ (20) \end{array}$$

LCCP($\mathcal{R}$) is *empty* and LCCP($E, \mathcal{R}$) consists of two *infeasible* LCCPs

$$\pi^{\text{LCCP}}_{\overrightarrow{(19)},(20)} : \langle \mathsf{g}(x), \mathsf{b} \rangle \Leftarrow \mathsf{a} = \mathsf{f}(x,x)$$
$$\pi^{\text{LCCP}}_{\overleftarrow{(19)},(20)} : \langle \mathsf{g}(x), \mathsf{a} \rangle \Leftarrow \mathsf{b} = \mathsf{f}(x,x)$$

hence trivially $\widetilde{\downarrow}_{\mathcal{R},E}$ -joinable.

Since $\mathcal{R}$ is *E-terminating* [Luc25, Example 5.17], it is *E-confluent*

DCPs in proofs of non-E-confluence

Consider again the ETRS $\mathcal{R}$ [Luc24, Example 14]:

$$\begin{array}{rcll} \mathsf{b} & = & \mathsf{f}(\mathsf{a}) & (21) \\ \mathsf{a} & = & \mathsf{c} & (22) \\ \mathsf{c} & \to & \mathsf{d} & (23) \\ \mathsf{b} & \to & \mathsf{d} & (24) \end{array}$$

All pairs in LCCP($\mathcal{R}$) are joinable. But

$$\pi^{\text{DCP}}_{(24)} : \langle \mathsf{d}, x' \rangle \Leftarrow x = \mathsf{b}, x \xrightarrow{>\Lambda} x'$$

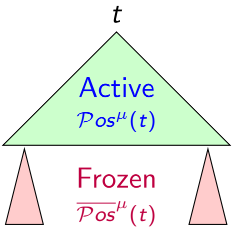
is not $\widetilde{\downarrow}_{\mathcal{R}/E}$ -joinable: $\sigma = \{x \mapsto \mathsf{f}(\mathsf{c}), x' \mapsto \mathsf{f}(\mathsf{d})\}$ satisfies $x = \mathsf{b}, x \xrightarrow{>\Lambda} x'$, but d and $\sigma(x') = \mathsf{f}(\mathsf{d})$ are not $\mathcal{R}/E$-joinable.

Thus, $\mathcal{R}$ is *not E-confluent*

Application to conditional systems

An *EGTRS* is a tuple $\mathcal{R} = (\mathcal{F}, \Pi, \mu, E, H, R)$ where

- $\mathcal{F}$ is a signature of function symbols,
- Π is a signature of predicate symbols with $=, \to, \to^* \in \Pi$,
- μ is a *replacement map* for $\mathcal{F}$, decomposing positions as follows:



Besides,

- E is a set of conditional equations $s = t \Leftarrow c$, for terms s and t;
- H is a set of definite Horn clauses $A \Leftarrow c$ where $A = P(t_1, \ldots, t_n)$ for some terms $t_1, \ldots, t_n$, $n \geq 0$, is such that $P \notin \{=, \to, \to^*\}$; and
- R is a set of conditional rules $\ell \to r \Leftarrow c$ for terms $\ell \notin \mathcal{X}$ and r.

where c is a sequence of atoms.

$$
\begin{array}{rcll}
xs ++ (ys ++ zs) & = & (xs ++ ys) ++ zs & (25)\\
\mathsf{Nat}(0) & & & (26)\\
\mathsf{Nat}(\mathsf{s}(n)) & \Leftarrow & \mathsf{Nat}(n) & (27)\\
x \approx y & \Leftarrow & x \to^{*} y & (28)\\
0 + n & \to & n & (29)\\
\mathsf{s}(m) + n & \to & \mathsf{s}(m + n) & (30)\\
\mathsf{sum}(m) & \to & n \Leftarrow m \approx n, \mathsf{Nat}(n) & (31)\\
\mathsf{sum}(ms) & \to & m + n & \\
 & \Leftarrow & ms \approx m ++ ns, \mathsf{Nat}(m), \mathsf{sum}(ns) \approx n & (32)
\end{array}
$$

Computational relations are defined by deduction of atoms in FO-theories:

$$\begin{array}{rcl} \mathsf{Th}_E & \vdash & s = t \\ \mathsf{Th}_{\mathcal{R}} & \vdash & s \to t \\ \mathsf{Th}_{\mathcal{R},E} & \vdash & s \stackrel{ps}{\to} t \\ \mathsf{Th}_{\mathcal{R}/E} & \vdash & s \stackrel{rm}{\to} t \end{array}$$

Each theory is obtained from generic sentences:

Label	Purpose	Label	Purpose
$(\mathrm{Rf})^{\bowtie}$	$\bowtie$ is reflexive	$(\mathrm{Tr})^{\bowtie}$	$\bowtie$ is transitive
$(\mathrm{Sy})^{\bowtie}$	$\bowtie$ is symmetric	$(\mathrm{Co})^{\bowtie,\bowtie^*}$	$\bowtie$ and then $\bowtie^*$ is in $\bowtie^*$
$(\mathrm{Pr})^{\bowtie}_{f,i}$	$\bowtie$ propagated in terms	$(\mathrm{HC})_{A\Leftarrow A_1,\ldots,A_n}$	Clause as sentence

$(\mathrm{R,E})_{\ell\to r\Leftarrow A_1,\ldots,A_n}$ Peterson & Stickel:

$$(\forall x,\vec{x})\ x = \ell \wedge A_1 \wedge \cdots \wedge A_n \Rightarrow x \stackrel{ps}{\to} r$$

$(\mathrm{R/E})$ Rewriting modulo:

$$(\forall x, x', y, y')\ x = x' \wedge x' \to y' \wedge y' = y \Rightarrow x \stackrel{rm}{\to} y$$

Extending J&K'86 abstract approach for ETRSs to EGTRSs?

Abstract reduction:	$\vdash\!\dashv_{\mathsf{E}}$	$\sim_{\mathsf{E}}$	$\to_{\mathsf{R}}$	$\to_{\mathsf{R}^{\mathsf{E}}}$	$\to_{\mathsf{R}/\mathsf{E}}$
Application to ETRSs $\mathcal{R}$:	$\to_{\underset{E}{\leftrightarrow}}$	$\to^*_{\underset{E}{\leftrightarrow}}$ is $=_E$	$\to_{\mathcal{R}}$	$\to_{\mathcal{R},E}$	$\to_{\mathcal{R}/E}$

Problem: (J&K2) fails for EGTRSs: $\to_{\mathcal{R}/E}$ is not $=_E \circ \to_{\mathcal{R}} \circ =_E$

$\mathsf{a} = \mathsf{b}$ $\qquad$ $\mathsf{a} \to \mathsf{c}$ $\qquad$ $\mathsf{a} \to \mathsf{d} \Leftarrow \mathsf{b} \to^* \mathsf{c}$

We have

- $\to_{\mathcal{R}} = \{(\mathsf{a},\mathsf{c})\}$ and
- $(=_E \circ \to_{\mathcal{R}} \circ =_E) = \{(\mathsf{a},\mathsf{c}),(\mathsf{b},\mathsf{c})\}$, but
- $\to_{\mathcal{R}/E} = \{(\mathsf{a},\mathsf{c}),(\mathsf{b},\mathsf{c}),(\mathsf{a},\mathsf{d}),(\mathsf{b},\mathsf{d})\}$.

Noticed by Meseguer [Mes17, Section 4.3]. His solution: $\mathsf{R} = \mathsf{R}^{\mathsf{E}} = \to_{\mathcal{R},E}$

Only ${}_{\mathcal{R},E}\!\leftarrow \circ \to_{\mathcal{R},E}$ and ${}_{\mathcal{R},E}\!\leftarrow \circ \vdash\!\dashv_E$ peaks are considered

E-confluence analysis based on *ECCPs*

In the *CR-theory* $\overline{\mathcal{R}^{\rm CR}}$ of an EGTRS,

goals $s \to t$ (resp $s \to^* t$)
in conditions of rules or Horn clauses
are *always* evaluated using $\to_{\mathcal{R}/E}$ (resp. $\to^*_{\mathcal{R}/E}$).

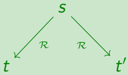
Then,

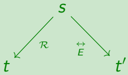
- $\overline{\mathcal{R}^{\rm CR}} \vdash s = t$ and $\overline{\mathcal{R}^{\rm CR}} \vdash s \stackrel{rm}{\to} t$ remain as $s =_E t$ and $s \to_{\mathcal{R}/E} t$.
- $\overline{\mathcal{R}^{\rm CR}} \vdash s \to t$ is denoted now as $s \to_{\mathcal{R}^{rm}} t$.
- $\overline{\mathcal{R}^{\rm CR}} \vdash s \stackrel{ps}{\to} t$ is denoted now as $s \to_{\mathcal{R}^{rm},E} t$.

Use of Jouannaud & Kirchner's framework with EGTRSs

Abstract reduction:	$\vdash\!\dashv_{\rm E}$	$\sim_{\rm E}$	$\to_{\rm R}$	$\to_{\rm R^E}$	$\to_{\rm R/E}$
Application to EGTRSs $\mathcal{R}$:	$\to_{\overset{\leftrightarrow}{E}}$	$=_E$	$\to_{\mathcal{R}^{rm}}$	$\to_{\mathcal{R}^{rm},E}$	$\to_{\mathcal{R}/E}$

Consider $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in \mathcal{R}^{rm}$ and $\beta : \lambda \to \rho \Leftarrow \gamma \in \overleftrightarrow{E}$

For r-peaks 

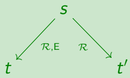
and c-peaks 

with $\widetilde{\downarrow}_{\mathcal{R}^{rm}}$

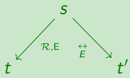
Set of GCPs	Label	Structure	Notes
$\mathrm{CCP}(\mathcal{R})$	$\pi_{\alpha,p,\alpha'}$	$\langle \theta(\ell)[\theta(r')]_p, \theta(r)\rangle \Leftarrow \theta(c), \theta(c')$	$p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\ell)$, $\ell\vert_p =^?_\theta \ell'$
$\mathrm{CVP}^{\to}(\mathcal{R})$	$\pi^{\to}_{\alpha,x,p}$	$\langle \ell[x']_p, r\rangle \Leftarrow x \to x', c$	$x \in \mathcal{V}ar^{\mu}(\ell)$, $p \in \mathcal{P}os^{\mu}_x(\ell)$
$\mathrm{CCP}(E,\mathcal{R})$	$\pi_{\beta,p,\alpha}$	$\langle \theta(\lambda)[\theta(r)]_p, \theta(\rho)\rangle \Leftarrow \theta(c), \theta(\gamma)$	$p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\lambda)$, $\lambda\vert_p =^?_\theta \ell$
$\mathrm{CCP}(\mathcal{R},E)$	$\pi_{\alpha,p,\beta}$	$\langle \theta(\ell)[\theta(\rho)]_p, \theta(r)\rangle \Leftarrow \theta(c), \theta(\gamma)$	$p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\lambda)$, $\ell\vert_p =^?_\theta \lambda$
$\mathrm{CVP}^{\to}(E)$	$\pi^{\to}_{\beta,x,p}$	$\langle \lambda[x']_p, \rho\rangle \Leftarrow x \to x', \gamma$	$x \in \mathcal{V}ar^{\mu}(\lambda)$, $p \in \mathcal{P}os^{\mu}_x(\lambda)$
$\mathrm{CVP}^{\vdash\!\dashv}(\mathcal{R})$	$\pi^{\vdash\!\dashv}_{\alpha,x,p}$	$\langle \ell[x']_p, r\rangle \Leftarrow x \vdash\!\dashv x', c$	$x \in \mathcal{V}ar^{\mu}(\ell)$, $p \in \mathcal{P}os^{\mu}_x(\ell)$

Consider $\alpha : \ell \to r \Leftarrow c, \alpha' : \ell' \to r' \Leftarrow c' \in \mathcal{R}^{rm}$ and $\beta : \lambda \to \rho \Leftarrow \gamma \in \overleftrightarrow{E}$

For PS-r-peaks



and PS-c-peaks



with $\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$

Set of GCPs	Label	Structure	Notes
$\mathsf{LCCP}(\mathcal{R})$	$\pi^{\mathrm{LCCP}}_{\alpha,p,\alpha'}$	$\langle \ell[r']_p, r\rangle \Leftarrow \ell\vert_p = \ell', c, c'$	$p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\ell)$
$\mathsf{CVP}^{\overset{ps}{\to}}(\mathcal{R})$	$\pi^{\overset{ps}{\to}}_{\alpha,x,p}$	$\langle \ell[x']_p, r\rangle \Leftarrow x \overset{ps}{\to} x', c$	$x \in \mathcal{V}ar^{\mu}(\ell)$, $p \in \mathcal{P}os^{\mu}_{x}(\ell)$
$\mathsf{DCP}(\mathcal{R})$	$\pi^{\mathrm{DCP}}_{\alpha}$	$\langle r, x'\rangle \Leftarrow x = \ell, x \overset{>\Lambda}{\longrightarrow} x', c$	
$\mathsf{LCCP}(E,\mathcal{R})$	$\pi^{\mathrm{LCCP}}_{\beta,p,\alpha}$	$\langle \lambda[r]_p, \rho\rangle \Leftarrow \lambda\vert_p = \ell, c, \gamma'$	$p \in \mathcal{P}os^{\mu}_{\mathcal{F}}(\lambda)$
$\mathsf{CVP}^{\overset{ps}{\to}}(E)$	$\pi^{\overset{ps}{\to}}_{\beta,x,p}$	$\langle \lambda[x']_p, \rho\rangle \Leftarrow x \overset{ps}{\to} x', \gamma$	$x \in \mathcal{V}ar^{\mu}(\lambda)$, $p \in \mathcal{P}os^{\mu}_{x}(\lambda)$

A rule $\ell \to r \Leftarrow c$ is

- *μ-left-linear* if no active variable occurs twice in ℓ
- *μ-left-homogeneous* if no active variable in ℓ is frozen in ℓ
- *μ-compatible* if no active variable in ℓ is frozen in r and no active variable in ℓ occurs in c

Joinability of CVPs for *conditional* rules $\alpha \in \mathcal{R}^{rm}$ and $\beta \in \overleftrightarrow{E}$

If α and β are left μ-homogeneous and μ-compatible, then

CVP	Joinable with
$\pi^{\to}_{\alpha,x,p}$	$\downarrow_{\mathcal{R}^{rm}}$
$\pi^{\to}_{\beta,x,p}$	$\widetilde{\downarrow}_{\mathcal{R}^{rm}}$
$\pi^{\vdash\!\dashv}_{\alpha,x,p}$	$\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$ ($\widetilde{\downarrow}_{\mathcal{R}^{rm}}$, if α is μ-left-linear)
$\pi^{\overset{ps}{\to}}_{\alpha,x,p}$	$\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$

Theorem (E-Confluence of EGTRSs $\mathcal{R}$ by $\widetilde{\downarrow}_{\mathcal{R}^{rm}}$ -joinability)

$\mathcal{R}$ is E-confluent if it is E-terminating and one of 1 – 3 holds:

1. *all conditional pairs in the following set are $\widetilde{\downarrow}_{\mathcal{R}^{rm}}$ -joinable:*

   $$\mathsf{CCP}(\mathcal{R}) \cup \mathsf{CVP}^{\rightarrow}(\mathcal{R}) \cup \mathsf{CCP}(E,\mathcal{R}) \cup \mathsf{CCP}(\mathcal{R},E) \cup \mathsf{CVP}^{\rightarrow}(E) \cup \mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})$$

2. *it is μ-left-homogeneous and μ-compatible, and all conditional pairs in the following set are $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable:*

   $$\mathsf{CCP}(\mathcal{R}) \cup \mathsf{CCP}(E,\mathcal{R}) \cup \mathsf{CCP}(\mathcal{R},E) \cup \mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})$$

3. *it is μ-left-linear, μ-left-homogeneous and μ-compatible, and all conditional pairs in the following set are $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable:*

   $$\mathsf{CCP}(\mathcal{R}) \cup \mathsf{CCP}(E,\mathcal{R}) \cup \mathsf{CCP}(\mathcal{R},E)$$

Theorem (Non-E-confluence)

If there is a non-$\widetilde{\downarrow}_{\mathcal{R}/E}$ -joinable $\pi \in \mathsf{CCP}(\mathcal{R}) \cup \mathsf{CVP}^{\rightarrow}(\mathcal{R})$, then $\mathcal{R}$ is not E-confluent

The "sum-of-nats" EGTRS $\mathcal{R}$ is E-confluent

1. $\mathcal{R}$ is μ-left-linear (hence μ-left-homogeneous) and μ-compatible.
2. The only *proper* CCP $\pi_{(33),\Lambda,(34)}$, where

$$\begin{array}{rcll} \mathsf{sum}(m) & \to & n \Leftarrow m \approx n, \mathsf{Nat}(n) & (33) \\ \mathsf{sum}(ms) & \to & m + n & \\ & \Leftarrow & ms \approx m +\!\!+\, ns, \mathsf{Nat}(m), \mathsf{sum}(ns) \approx n & (34) \end{array}$$

   is *infeasible*, hence $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable:

$$\langle m + n, n' \rangle \Leftarrow ms \approx_{rm} n', \mathsf{Nat}(n'), ms \approx_{rm} m +\!\!+\, ns, \mathsf{Nat}(m), \mathsf{sum}(ns) \approx_{rm} n$$

3. *Improper* CCPs $\pi^{\mathrm{I}}_{(33)}$ and $\pi^{\mathrm{I}}_{(34)}$ are proved $\widetilde{\downarrow}_{\mathcal{R}}$ -joinable by induction
4. CCP$(E, \mathcal{R})$ and CCP$(\mathcal{R}, E)$ are *empty*
5. It is not difficult to see that $\mathcal{R}$ is E-terminating.

Thus, $\mathcal{R}$ is E-confluent

Theorem (Confluence of EGTRSs $\mathcal{R}$ by $\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$ -joinability)

$\mathcal{R}$ is E-confluent if it is E-terminating and one of 1 or 2 holds:

1. *all conditional pairs in the following set are $\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$ -joinable:*

$$\mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}^{\overset{ps}{\to}}(\mathcal{R}) \cup \mathsf{LCCP}(E,\mathcal{R}) \cup \mathsf{CVP}^{\overset{ps}{\to}}(E)$$

2. *$R \cup \overset{\leftrightarrow}{E}$ is μ-left-homogeneous and μ-compatible and all conditional pairs in the following set are $\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$ -joinable:*

$$\mathsf{LCCP}(\mathcal{R}) \cup \mathsf{LCCP}(E,\mathcal{R})$$

Theorem (Non-E-confluence)

If there is a non- $\widetilde{\downarrow}_{\mathcal{R}/E}$ -joinable $\pi \in \mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}^{\overset{ps}{\to}}(\mathcal{R}) \cup \mathsf{DCP}(\mathcal{R})$*, then $\mathcal{R}$ is not E-confluent*

Subsumes non- $\widetilde{\downarrow}_{\mathcal{R}/E}$ -joinability of $\pi \in \mathsf{CCP}(\mathcal{R}) \cup \mathsf{CVP}^{\to}(\mathcal{R})$

Consider the following EGTRS $\mathcal{R}$ where $R \cup \overleftrightarrow{E}$ is μ-left-linear and μ-compatible:

$$
\begin{array}{rclr}
\mathsf{a} & = & \mathsf{b} & (35) \\
\mathsf{a} & \to & \mathsf{c} & (36)
\end{array}
\qquad
\begin{array}{rclr}
\mathsf{a} & \to & \mathsf{d} \Leftarrow \mathsf{b} \to^{*} \mathsf{c} & (37) \\
\mathsf{c} & \to & \mathsf{d} & (38)
\end{array}
$$

The CCP $\pi_{\overrightarrow{(35)},\Lambda,(36)} : \langle \mathsf{c}, \mathsf{b} \rangle \in \mathrm{CCP}(E, \mathcal{R})$ is *not $\widetilde{\downarrow}_{\mathcal{R}^{rm}}$ -joinable*

However, $\pi^{\mathrm{LCCP}}_{\overrightarrow{(35)},\Lambda,(36)} : \langle \mathsf{c}, \mathsf{b} \rangle \Leftarrow \mathsf{a} = \mathsf{a} \in \mathrm{LCCP}(E, \mathcal{R})$ is *$\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$ -joinable*

All pairs in $\mathrm{LCCP}(\mathcal{R}) \cup \mathrm{LCCP}(E, \mathcal{R})$ are $\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$ -joinable

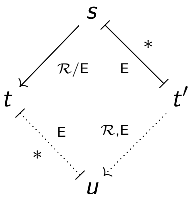
E-termination of $\mathcal{R}$ is proved as explained above [Luc25, Example 11.6]

Thus, $\mathcal{R}$ is *E-confluent*

Related work

Durán and Meseguer prove *E-confluence of conditional rewrite theories* $\mathcal{R} = (\mathcal{F}, E, R)$ such that [DM12, Theorem 2]:

1. E is a set of *linear* and *regular* unconditional equations;
2. R is *strongly E-coherent* [DM12, page 819]:



3. the rules in R are *strongly deterministic*; and
4. $\mathcal{R}$ is *quasi-decreasing*.

E-confluence is *characterized* as the $\widetilde{\downarrow}_{\mathcal{R},E}$-joinability of each ECCP

$$\langle \theta(\ell)[\theta(r')]_p, \theta(r)\rangle \Leftarrow \theta(c), \theta(c')$$

where $p \in \mathcal{P}os_{\mathcal{F}}(\ell)$ and $\ell|_p =^{?}_{E,\theta} \ell'$

The *E-confluence* of

$$\begin{array}{rclr} a & = & b & (39) \\ a & \to & c & (40) \end{array} \qquad \begin{array}{rclr} a & \to & d \Leftarrow b \to^* c & (41) \\ c & \to & d & (42) \end{array}$$

cannot be proved as rule (41) is *not* strongly deterministic (c is reducible)

The *non-E-confluence* of $\mathcal{R}$ in [Luc24, Example 14]:

$$\begin{array}{rcl} b & = & f(a) \\ a & = & c \end{array} \qquad \begin{array}{rcl} c & \to & d \\ b & \to & d \end{array}$$

cannot be proved as $\mathcal{R}$ is *not strongly E-coherent*. We have

$$f(d) \;{}_{\mathcal{R}/E}\!\leftarrow \quad b \quad =_E b$$

but $b \to_{\mathcal{R},E} d$ is the only $\to_{\mathcal{R},E}$-step on b, and $f(d) \neq_E d$.

Note: the rules define *no ECCP*

Conclusions and future work

E-confluence of E-terminating EGTRSs $\mathcal{R}$ can be *proved* by checking:

- $\widetilde{\downarrow}_{\mathcal{R}^{rm}}$-joinability of
  - *Conditional Critical Pairs* in $\mathsf{CCP}(\mathcal{R})$, $\mathsf{CCP}(E,\mathcal{R})$, $\mathsf{CCP}(\mathcal{R},E)$, and
  - *Conditional Variable Pairs* in $\mathsf{CVP}^{\rightarrow}(\mathcal{R})$, $\mathsf{CVP}^{\rightarrow}(E)$, and $\mathsf{CVP}^{\vdash\!\dashv}(\mathcal{R})$
- $\widetilde{\downarrow}_{\mathcal{R}^{rm},E}$-joinability of
  - *Logic-based Conditional Critical Pairs* in $\mathsf{LCCP}(\mathcal{R})$ and $\mathsf{LCCP}(E,\mathcal{R})$, and
  - *Conditional Variable Pairs* in $\mathsf{CVP}^{\overset{ps}{\rightarrow}}(\mathcal{R})$ and $\mathsf{CVP}^{\overset{ps}{\rightarrow}}(E)$

Simplifications possible depending on the *structure of equations and rules*

Our methods apply to *ETRSs* and *Conditional Rewrite Theories* as particular cases of EGTRSs

Non-E-confluence of EGTRSs can be proved as the non-$\widetilde{\downarrow}_{\mathcal{R}/E}$-joinability of some conditional pair in $\mathsf{LCCP}(\mathcal{R}) \cup \mathsf{CVP}^{\overset{ps}{\rightarrow}}(\mathcal{R}) \cup \mathsf{DCP}(\mathcal{R})$

Our examples would *not* be handled by previous works [JK86, DM12].

Implementation

Envisaged implementation in our confluence tool CONFident, including:

- Improving MU-TERM to prove *E*-termination of EGTRSs
- Improving infChecker to deal with (in)feasibility proofs with EGTRSs

Theoretical developments

Conditions to use $\widetilde{\downarrow}_{\mathcal{R},E}$-joinability of CCPs instead of ECCPs or LCCPs

Develop methods to prove *E*-termination of EGTRSs

Use abstract results of *E*-confluence not requiring *E*-termination

Extension to other rewriting-based frameworks

- Logically-Constrained Term Rewriting Systems,
- Higher-Order Rewriting,
- Nominal Rewriting

Application to Maude

Thanks!

Confluence of Conditional Rewriting Modulo
– Invited talk –

Salvador Lucas

DSIC & VRAIN, Universitat Politècnica de València, Spain

14th International Workshop on Confluence

IWC 2025

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