universität innsbruck

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Improving Confluence Analysis for LCTRSs

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University of Innsbruck

Outline

compute $\sum_{i=1}^{n} i$ for natural number n

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Example

- term rewrite system (TRS)

$$\mathsf{sum}(0) \to 0 \qquad \mathsf{add}(0, y) \to y$$
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- $\mathcal{R}_{\text{ca}}$ is set of calculation rules and $\mathcal{R}_{\text{rc}} = \mathcal{R} \cup \mathcal{R}_{\text{ca}}$

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notation: $\sigma \vDash \rho$

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① position p in s

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such that $s|_p = \ell\sigma$, $t = s[r\sigma]_p$ and $\sigma \vDash \ell \to r\ [\varphi]$

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  1. position 1
  2. calculation rule $x_1 - x_2 \to y\ [y = x_1 - x_2]$
  3. substitution $\sigma = \{x_1 \mapsto 3, x_2 \mapsto 1, y \mapsto 2\}$

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- $\psi = \mathcal{EC}_{\rho_1} \wedge \mathcal{EC}_{\rho_2}$ where $\mathcal{EC}_\rho$ with $\rho \colon \ell \to r \ [\varphi]$ abbreviates $\bigwedge \{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho)\}$

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  5. if $p = \epsilon$ then ρ_1 and ρ_2 are not variants or $\mathcal{V}\mathsf{ar}(r_1) \nsubseteq \mathcal{V}\mathsf{ar}(\ell_1)$
- $\ell_2\sigma[r_1\sigma]_p \approx r_2\sigma \;\, [\varphi_1\sigma \wedge \varphi_2\sigma \wedge \psi\sigma]$ is induced constrained critical pair
- $\mathcal{EV}\mathsf{ar}(\ell \to r \;\, [\varphi]) = \mathcal{V}\mathsf{ar}(r) \setminus (\mathcal{V}\mathsf{ar}(\ell) \cup \mathcal{V}\mathsf{ar}(\varphi))$ is set of extra variables
- $\psi = \mathcal{EC}_{\rho_1} \wedge \mathcal{EC}_{\rho_2}$ where $\mathcal{EC}_\rho$ with $\rho\colon \ell \to r \;\, [\varphi]$ abbreviates $\bigwedge \{x = x \mid x \in \mathcal{EV}\mathsf{ar}(\rho)\}$
- substitution σ respects constraint φ $(\sigma \vDash \varphi)$ if $\sigma(x) \in \mathcal{V}\mathsf{al}$ for $x \in \mathcal{V}\mathsf{ar}(\varphi)$ and $[\![\varphi\sigma]\!] = \top$

Definitions

- overlap of LCTRS $\mathcal{R}$ is triple $\langle \rho_1, p, \rho_2 \rangle$ such that
  1. $\rho_1 \colon \ell_1 \to r_1 \ [\varphi_1]$ and $\rho_2 \colon \ell_2 \to r_2 \ [\varphi_2]$ are variable-disjoint variants of rules in $\mathcal{R}_{\mathsf{rc}}$
  2. $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(\ell_2)$
  3. ℓ_1 and $\ell_2|_p$ unify with mgu σ such that $\sigma(x) \in \mathcal{V}\mathsf{al} \cup \mathcal{V}$ for all $x \in \mathcal{LV}\mathsf{ar}(\rho_1) \cup \mathcal{LV}\mathsf{ar}(\rho_2)$
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- constrained equation $s \approx t \ [\varphi]$ is trivial if $s\sigma = t\sigma$ for every substitution σ with $\sigma \vDash \varphi$

Outline

Confluence Methods for TRSs

joinable critical pairs for terminating systems

orthogonality

Confluence Methods for TRSs

- critical pair closing systems
- decreasing diagrams
- development closed critical pairs
- discrimination pairs
- joinable critical pairs for terminating systems
- orthogonality
- parallel closed critical pairs
- parallel critical pairs
- reduction method
- redundant rules
- rule labeling
- simultaneous critical pairs
- source labeling
- strongly closed critical pairs
- tree automata
- weak orthogonality
- Z property
- ...

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Definition (Transformation)

LCTRS $\mathcal{R}$ is transformed into TRS $\overline{\mathcal{R}}$ consisting of

$$\ell\tau \to r\tau$$

for all $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}_{\mathsf{rc}}$ and substitutions τ with $\tau \vDash \rho$ and $\mathcal{D}\mathsf{om}(\tau) = \mathcal{LV}\mathsf{ar}(\rho)$

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Corollary

LCTRS $\mathcal{R}$ is confluent $\iff$ TRS $\overline{\mathcal{R}}$ is confluent

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advanced confluence criteria require rewriting of constrained terms and equations

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Example

- TRS $\mathcal{R} = \{\mathsf{f}(\mathsf{f}(x)) \to x,\ \mathsf{f}(x) \to \mathsf{f}(\mathsf{f}(x))\}$ has two non–trivial critical pairs

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$$\mathsf{f}(\mathsf{f}(\mathsf{f}(x))) \approx x \qquad\qquad x \approx \mathsf{f}(\mathsf{f}(\mathsf{f}(x)))$$

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- adding rule $\mathsf{f}(x) \to x$ results in four new critical pairs
- resulting TRS is development−closed

Definitions

- constrained rewrite rule $\rho\colon \ell \to r\ [\varphi] \in \mathcal{R}$ is redundant if

$$\ell \approx r\ [\varphi \wedge \mathcal{EC}_\rho] \xrightarrow{\sim}{}^{*}_{\mathcal{R}\setminus\{\rho\},\geqslant 1} \ell' \approx r'\ [\psi]$$

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Theorem

if set of constrained rules $\mathcal{S}$ is redundant in LCTRS $\mathcal{R}$ and $\mathcal{R} \simeq \mathcal{S}$ then

$$\mathcal{R} \text{ is confluent} \quad \Longleftrightarrow \quad \mathcal{R} \cup \mathcal{S} \text{ is confluent}$$

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if for every rule $\rho\colon \ell \to r\ [\varphi] \in \mathcal{S}$

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for some trivial $\ell' \approx r'\ [\psi]$ and $\mathcal{R} \simeq \mathcal{S}$ then

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LCTRS $\mathcal{R}$ over theory Ints

$$\mathsf{f}(x,y) \xrightarrow{\alpha} x + y \quad [x > 0] \qquad \mathsf{f}(x,y) \xrightarrow{\beta} \mathsf{d}(x,y)\ [x = 2 \cdot y \wedge y > 0]$$

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has two constrained critical pairs with constraint $\varphi = (x = 2 \cdot y \wedge y > 0)$

$$x + y \approx \mathsf{d}(x,y)\ [x > 0 \wedge \varphi] \qquad \mathsf{d}(x,y) \approx x + y\ [\varphi \wedge x > 0]$$

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&\xleftrightarrow{\sim}_{\mathcal{R}\setminus\{\beta\},>\epsilon} z \approx z' \ [\varphi \wedge z = x+y \wedge z' = y+x]
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$$\begin{aligned} \mathsf{f}(x,y) \approx \mathsf{d}(x,y) \; [\varphi] &\overset{\sim}{\longleftrightarrow}_{\mathcal{R}\setminus\{\beta\},>\epsilon} x+y \approx \mathsf{d}(x,y) \; [\varphi] \\ &\overset{\sim}{\longleftrightarrow}_{\mathcal{R}\setminus\{\beta\},>\epsilon} x+y \approx y+x \; [\varphi] \\ &\overset{\sim}{\longleftrightarrow}_{\mathcal{R}\setminus\{\beta\},>\epsilon} z \approx y+x \; [\varphi \wedge z = x+y] \\ &\overset{\sim}{\longleftrightarrow}_{\mathcal{R}\setminus\{\beta\},>\epsilon} z \approx z' \; [\varphi \wedge z = x+y \wedge z' = y+x] \end{aligned}$$

LCTRS $\mathcal{R} \setminus \{\beta\}$ is orthogonal

Example (cont'd)

LCTRS $\mathcal{R}$ over theory Ints

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LCTRS $\mathcal{R} \setminus \{\beta\}$ is orthogonal $\implies$ $\mathcal{R}$ confluent

Implementation Heuristics

① for $s \approx t \; [\varphi] \in \mathrm{CCP}(\mathcal{R})$ if

$$s \approx t \; [\varphi] \;\multimap\!\!\rightarrow^{2}_{\mathcal{R},\geqslant 1} \cdot \multimap\!\!\rightarrow^{2}_{\mathcal{R},\geqslant 2} \; u \approx v \; [\psi]$$

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Outline

1. Logically Constrained Rewrite Systems
2. Confluence Results
3. Redundant Rules
4. **Reduction Method**
5. Conclusion

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$\mathcal{R}\restriction_{\mathcal{C}} = \{\ell \to r \in \mathcal{R} \mid \mathcal{F}\mathsf{un}(\ell) \subseteq \mathcal{F}\mathsf{un}(\mathcal{C})\}$ for TRSs $\mathcal{R}$ and $\mathcal{C}$

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Corollary

if left–linear LCTRS $\mathcal{R}$ is convertible by LCTRS $\mathcal{C}$ and $\mathcal{R}\restriction_{\mathcal{C}}$ is simulated by $\mathcal{C}$ then

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Outline

development closed critical pairs

joinable critical pairs for terminating systems

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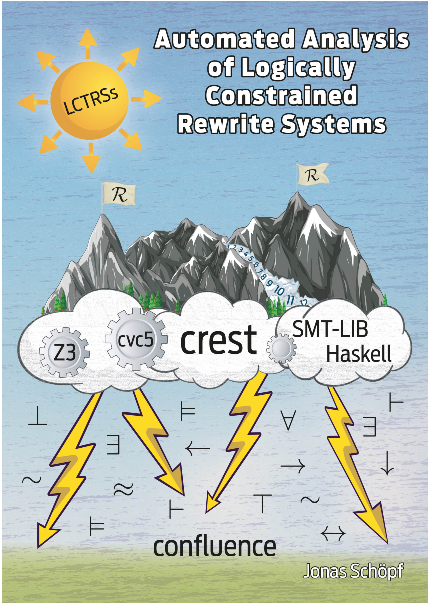
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