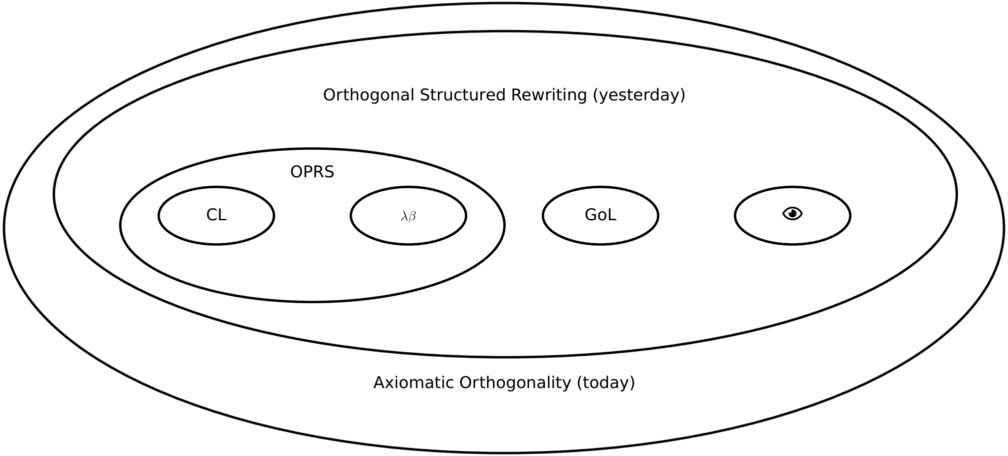
Redeeming Newman; orthogonality in rewriting
Past, present and future in a 1-algebraic setting

Vincent van Oostrom

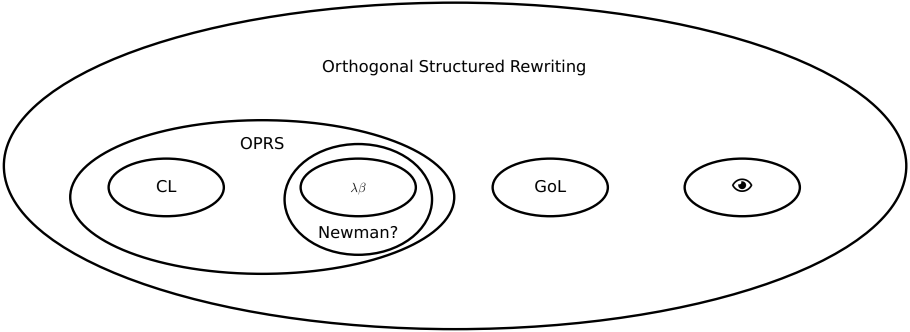
Part I: Axiomatic Orthogonality

Part II: Newman’s axiomatics

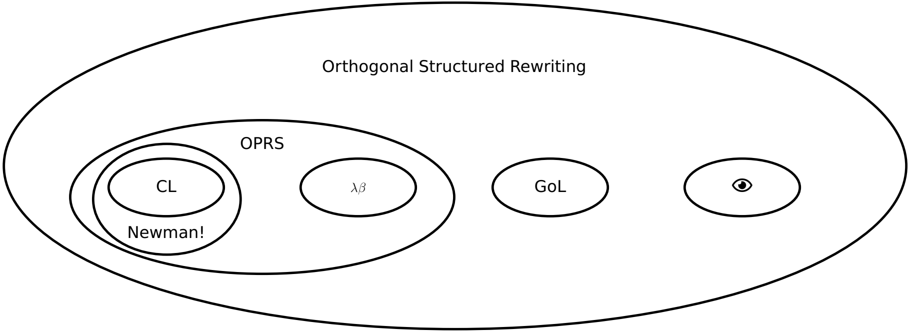
Orthogonality



Orthogonality



Orthogonality



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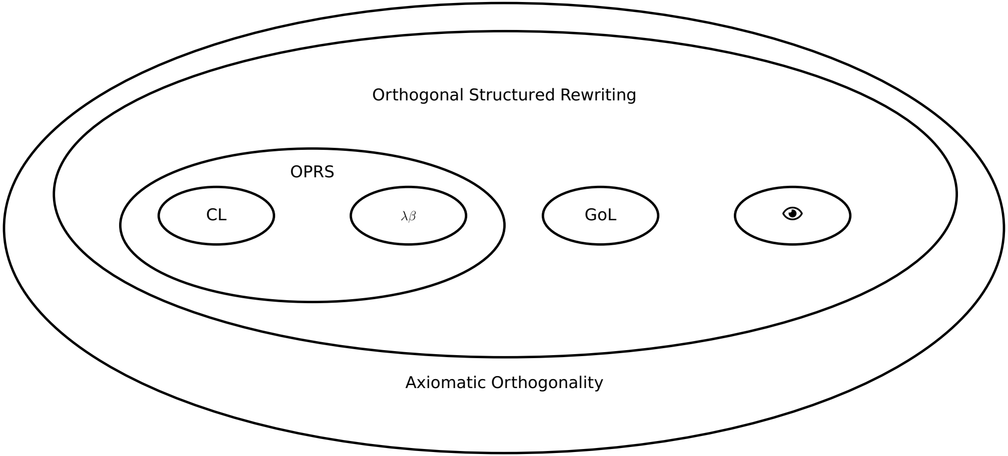

Theory of Orthogonality

Theory of Orthogonality (Terese 03)

- sequentialisation: $\to \subseteq \multimap \subseteq \twoheadrightarrow$
  (for some notion of parallel or multistep $\multimap$)

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- sequentialisation: $\rightarrow \subseteq \multimap\!\!\!\rightarrow \subseteq \twoheadrightarrow$
- confluence: $\multimap\!\!\!\rightarrow$ has the diamond property
  (for some notion of residuation $/$)

Theory of Orthogonality

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- sequentialisation: $\to \subseteq \mathrel{-\!\!\circ\!\!\to} \subseteq \twoheadrightarrow$
- confluence: $\mathrel{-\!\!\circ\!\!\to}$ has the diamond property
- orthogonal: tiling 3-peak of $\mathrel{-\!\!\circ\!\!\to}$-steps with diamonds yields a cube

  (entails co-initial reductions form semi-lattice; least upperbounds)

Theory of Orthogonality

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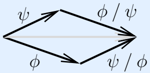
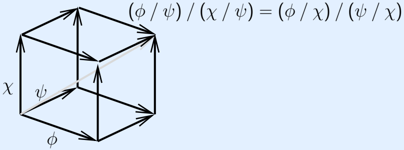
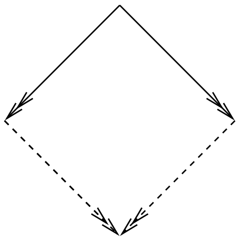


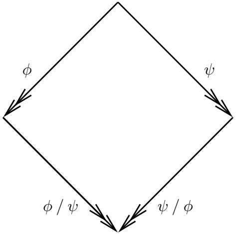


Confluence vs. orthogonality



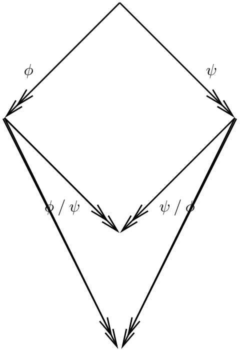
confluence, upperbound

Confluence vs. orthogonality



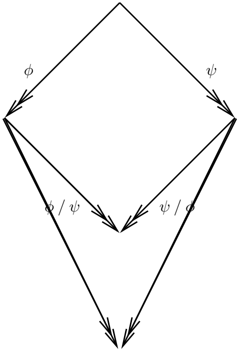
confluence, upperbound via witnessing residual function $/$

Confluence vs. orthogonality



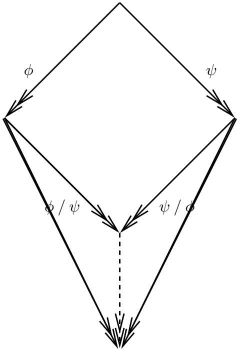
orthogonality, other upperbounds . . .

Upperbounds vs. least upperbounds



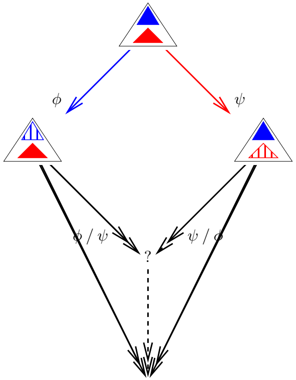
orthogonality, least among upperbounds?

Upperbounds vs. least upperbounds



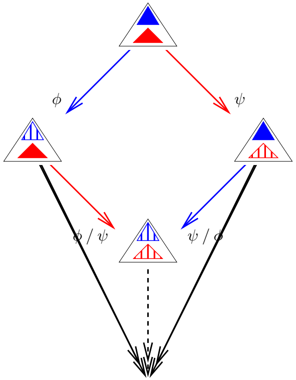
orthogonality, least upperbound

Upperbounds vs. least upperbounds



orthogonality, least upperbound doing work of both (Lévy; $I(IK)$)

Upperbounds vs. least upperbounds



orthogonality, least upperbound doing work of both in $I(IK)$

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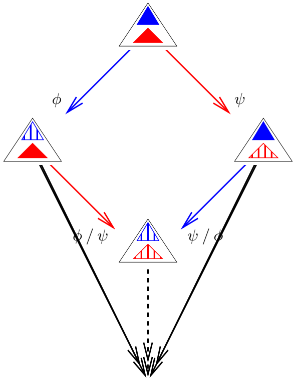

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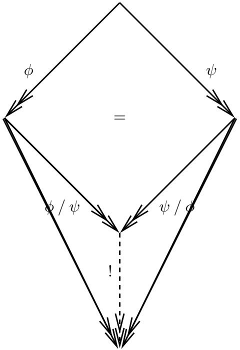
Diamond vs. cube



orthogonality, least upperbound w.r.t. notion of same work

Syntactic vs. axiomatic orthogonality?

OTRS (1990)

term rewrite system (TRS) is orthogonal if left-linear and no critical pairs

Syntactic vs. axiomatic orthogonality?

OTRS

term rewrite system (TRS) is orthogonal if left-linear and no critical pairs

Programme (since 1990s; Melliès, Khasidashvili, , . . .)

appropriate definitions of step and orthogonality axioms such that

- OTRS entails all steps are orthogonal to each other
- orthogonality axioms entail theory of orthogonality

Syntactic vs. axiomatic orthogonality?

Programme

appropriate definitions of step and orthogonality axioms such that

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Example

- for CL / OTRSs $\twoheadrightarrow\!\!\!\!\!\!\!\!\!+\!\!\!+$ and $\multimap\!\!\!\!\!\!\rightarrow$ are orthogonal (Huet & Lévy)

Syntactic vs. axiomatic orthogonality?

Programme

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- for CL / OTRSs $\mathrel{-\!\!\!\|\!\!\!\rightarrow}$ and $\mathrel{-\!\!\circ\!\!\rightarrow}$ are orthogonal (Huet & Lévy)
- braids and self-distributivity are orthogonal (Melliès, Schikora)

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- braids and self-distributivity are orthogonal (Melliès, Schikora)
- for $\lambda\beta$ / OPRSs $\mathrel{-\!\!\circ\!\!\rightarrow}$ is orthogonal (Lévy / Klop, Bruggink)

Syntactic vs. axiomatic orthogonality?

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appropriate definitions of step and orthogonality axioms such that

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- for CL / OTRSs $\mathrel{\longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\!\!\!\!+\;\;\;}$ and $\mathrel{-\!\!\circ\!\!\twoheadrightarrow}$ are orthogonal (Huet & Lévy)
- braids and self-distributivity are orthogonal (Melliès, Schikora)
- for $\lambda\beta$ / OPRSs $\mathrel{-\!\!\circ\!\!\twoheadrightarrow}$ is orthogonal (Lévy / Klop, Bruggink)
- ...

Combinatory Logic (CL) $\multimap\!\!\!\rightarrow$

named combinatory logic (CL) rules:

$$\begin{array}{rrcl} \iota(x): & Ix & \to & x \\ \kappa(x,y): & Kxy & \to & x \\ \varsigma(x,y,z): & Sxyz & \to & xz(yz) \end{array}$$

Definition

multi-step ARS $\multimap\!\!\!\rightarrow$:

- objects: terms over alphabet
- steps: terms over function symbols + rule names
- $\mathrm{src}(f(\vec{s})) = f(\mathrm{src}(\vec{s}))$ with f function symbol
  $\mathrm{src}(\varrho(\vec{s})) = l(\mathrm{src}(\vec{s}))$ with $\varrho(\vec{x})$ name of rule $l(\vec{x}) \to r(\vec{x})$

step ARS $\to$: restriction of $\multimap\!\!\!\rightarrow$ steps to exactly one rule name

$\iota(IK): I(IK) \multimap\!\!\!\rightarrow IK$ $\qquad$ $I(\iota(K)): I(IK) \multimap\!\!\!\rightarrow IK$

$I(IK): I(IK) \multimap\!\!\!\rightarrow I(IK)$ $\qquad$ $\iota(\iota(K)): I(IK) \multimap\!\!\!\rightarrow K$

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$\iota(IK) : I(IK) \to IK$ $\qquad$ $I(\iota(K)) : I(IK) \to IK$

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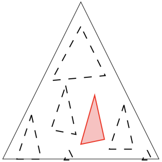
Combinatory Logic (CL) $\multimap\!\!\!\to$



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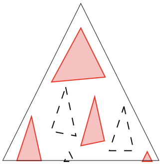
Combinatory Logic (CL) $-\!\!\circ\!\!\!\rightarrow$



multistep $-\!\!\circ\!\!\!\rightarrow$: contract some redex-patterns

$$\rightarrow \;\subseteq\; -\!\!\circ\!\!\!\rightarrow \;\subseteq\; \twoheadrightarrow$$

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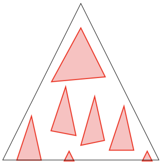
Combinatory Logic (CL) $\multimap\!\!\!\rightarrow$



full multistep $-\!\!\bullet\!\!\rightarrow$: contract all redex-patterns

$-\!\!\bullet\!\!\rightarrow \;\subseteq\; -\!\!\circ\!\!\rightarrow \;\subseteq\; \twoheadrightarrow$

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CL / OTRS residuation

Intuition

residual ϕ/ψ *of step* ϕ *after step* ψ*:*
what remains (to be done) of step ϕ *after doing* ψ*.*

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CL / OTRS residuation

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Example

residual of $I(\iota(K)) : I(IK) \multimap\!\!\!\!\to IK$ after
$\iota(IK) : I(IK) \multimap\!\!\!\!\to IK$?

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Example

residual of $I(\iota(K)) : \ I(IK) \mathrel{-\!\circ\!\!\to} IK$ after
$\iota(IK) : \ I(IK) \mathrel{-\!\circ\!\!\to} IK$?
$\iota(K) : \ IK \mathrel{-\!\circ\!\!\to} K$!
and conversely?
same (but now residual is blue!)

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CL / OTRS residuation

Intuition

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Example

residual of $SIK(IK) \multimap\!\!\to SIKK$ after
$SIK(IK) \multimap\!\!\to I(IK)(K(IK))$?

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Example

residual of $SIK(IK) \multimap SIKK$ after
$SIK(IK) \multimap I(IK)(K(IK))$?
$I(IK)(K(IK)) \multimap IK(KK)$!
and conversely?
$SIKK \multimap IK(KK)$!

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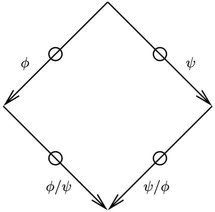

ϕ/ψ and ψ/ϕ: multisteps ending in same object

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CL / OTRS residuation

Definition

1-ra is rewrite system with 1-operations

- 1 the empty step for each object (doing nothing)
- / the residual map from pairs of (co-initial) steps to steps
- satisfying axioms

$$
\begin{array}{rcl}
\phi/\phi & = & 1 \\
\phi/1 & = & \phi \\
1/\phi & = & 1 \\
(\phi/\psi)/(\chi/\psi) & = & (\phi/\chi)/(\psi/\chi) \quad \text{(cube)}
\end{array}
$$

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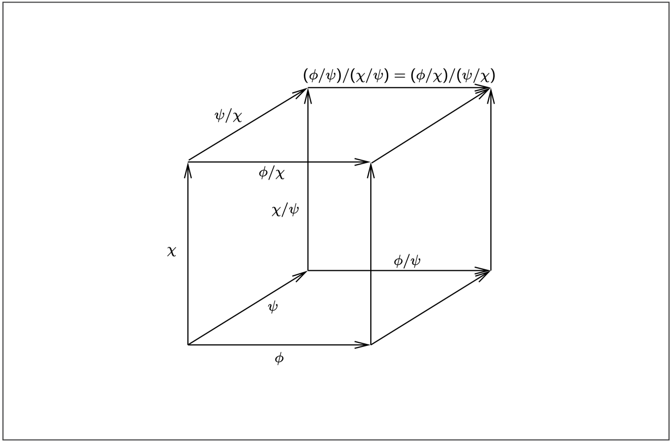
CL / OTRS residuation



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CL / OTRS residuation

Theorem

left-linear and non-overlapping TRS induces 1-ra *on multisteps* $\multimap\!\!\rightarrow$

- empty multisteps as units
- residual operation defined by induction on multisteps

$$
\begin{array}{rcl}
f(\phi_1,\ldots,\phi_n)/f(\psi_1,\ldots,\psi_n) & = & f(\phi_1/\psi_1,\ldots,\phi_n/\psi_n) \\
\varrho(\phi_1,\ldots,\phi_n)/l(\psi_1,\ldots,\psi_n) & = & \varrho(\phi_1/\psi_1,\ldots,\phi_n/\psi_n) \\
l(\phi_1,\ldots,\phi_n)/\varrho(\psi_1,\ldots,\psi_n) & = & r(\phi_1/\psi_1,\ldots,\phi_n/\psi_n) \\
\varrho(\phi_1,\ldots,\phi_n)/\varrho(\psi_1,\ldots,\psi_n) & = & r(\phi_1/\psi_1,\ldots,\phi_n/\psi_n)
\end{array}
$$

for every rule $\varrho(x_1,\ldots,x_n) : \ l(x_1,\ldots,x_n) \to r(x_1,\ldots,x_n)$

Example

- $I(\iota(K))/\iota(IK) = \iota(K)$
- $SIK(\iota(K))/\varsigma(I,K,IK) = I(\iota(K))(K(\iota(K)))$

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CL / OTRS residuation

Definition

1-rac is rewrite system with 1-operations

- 1 the empty step for each object (doing nothing)
- / the residual map from pairs of (co-initial) steps to steps
- · the composition map on composable steps
- satisfying axioms

$$
\begin{aligned}
\phi/\phi &= 1 \\
\phi/1 &= \phi \\
1/\phi &= 1 \\
(\phi/\psi)/(\chi/\psi) &= (\phi/\chi)/(\psi/\chi) \\
1 \cdot 1 &= 1 \\
\chi/(\phi \cdot \psi) &= (\chi/\phi)/\psi \\
(\phi \cdot \psi)/\chi &= (\phi/\chi) \cdot (\psi/(\chi/\phi))
\end{aligned}
$$

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Orthogonality: from steps to reductions

Lemma

1-*ra on* $\rightarrow$ *induces a* 1-*rac on* $\twoheadrightarrow$ *(by tiling)*

Orthogonality: from steps to reductions

Lemma

1-ra on $\to$ induces a 1-rac on $\twoheadrightarrow$ (by tiling)

Example

- $\langle \{0,1\}, 0, \dot{-} \rangle$ is a 1-ra, for $\dot{-}$ monus (cut-off subtraction)
- $\langle \mathbb{N}, 0, \dot{-}, + \rangle$ is a 1-rac (induced by 1-ra)

Orthogonality: from steps to reductions

Definition

$\phi \preceq \psi := (\phi \,/\, \psi = 1)$ is natural order on co-initial steps ϕ, ψ

Orthogonality: from steps to reductions

Definition

$\phi \preceq \psi := (\phi \,/\, \psi = 1)$ is natural order on co-initial steps ϕ, ψ

Theorem

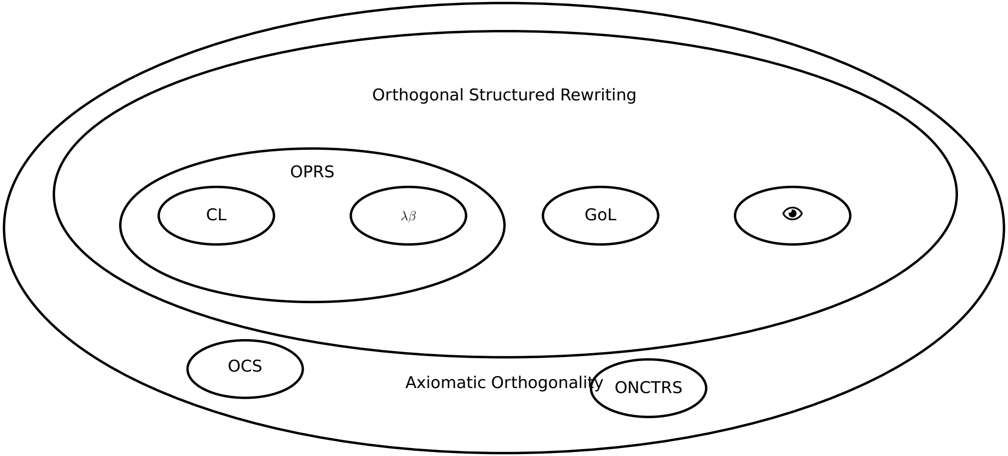
$\langle \rightarrow, 1, /, \cdot \rangle$ is a 1-rac whose natural order is a partial order
where $\phi \,/\, \psi := \phi'$ for every peak ϕ, ψ and its pushout valley ψ', ϕ'

iff

$\langle \rightarrow, 1, \cdot \rangle$ is a 1-monoid that is

- *left-cancellative (each χ is epi: for all ϕ, ψ, if $\chi \cdot \phi = \chi \cdot \psi$ then $\phi = \psi$)*
- *gaunt (isomorphisms are 1)*
- *has pushouts (lubs of peaks exist)*

Axiomatic Orthogonality; 1-ra(c)



Axiomatic Orthogonality; 1-ra(c)

Theorem

- *orthogonal context-sensitive TRS (OCS; Lucas) are orthogonal*
- *orthogonal normal CTRS (ONCTRS; Bergstra & Klop) are orthogonal*

Axiomatic Orthogonality; 1-ra(c)

Theorem

- *orthogonal context-sensitive TRS (OCS; Lucas) are orthogonal*
- *orthogonal normal CTRS (ONCTRS; Bergstra & Klop) are orthogonal*

Proof.

by simple adaptation of OTRS case, of multisteps and residuation $/$:

- OCS: all frozen arguments are terms
  (not steps; no redexes inside)
- ONCTRS: conditions must hold for sources of steps in arguments of rules
  (uses stability) □

Axiomatic Orthogonality; 1-ra(c)

Theorem

- *orthogonal context-sensitive TRS (OCS; Lucas) are orthogonal*
- *orthogonal normal CTRS (ONCTRS; Bergstra & Klop) are orthogonal*

Programme

investigate for structured rewrite systems $\mathcal{T}$ declared to be orthogonal in the literature, whether they have a natural underlying 1-ra (*e.g.*, $\twoheadrightarrow\!\!\!\!\!\!+\;_{\mathcal{T}}$)

Newman's axiomatic confluence result (1942)

Newman's Goal

axiomatise Church–Rosser's 1936 confluence proof for λ-calculus

Newman's axiomatic confluence result

Newman's Goal

axiomatise Church–Rosser's 1936 confluence proof for λ-calculus

Notations

- $\phi \mid \psi$ denotes set of ψ-derivates of ϕ
  (for co-initial ϕ, ψ; each ψ-derivate is step from target of ψ)

Newman's axiomatic confluence result

Newman's Goal

axiomatise Church–Rosser's 1936 confluence proof for λ-calculus

Notations

- $\phi \mid \psi$ denotes set of ψ-derivates of ϕ
- ${}^{\varrho}_{\varsigma'}\!\diamond^{\varsigma}_{\varrho'}$ denotes $\varrho, \varsigma, \varsigma', \varrho'$ form a diamond

Newman's axiomatic confluence result

Theorem

there are reductions ς', ϱ' *such that* ${}^{\varrho}_{\varsigma'}\diamond^{\varsigma}_{\varrho'}$, *and* $\Phi \mid (\varrho \cdot \varsigma') = \Phi \mid (\varsigma \cdot \varrho')$ *given co-initial reductions* ϱ, ς *and set of steps* Φ, *if for a predicate J:*

(Δ_1) $\phi \mid \psi = \emptyset$ *iff* $\phi = \psi$

(Δ_2) *if* $\phi \neq \psi$, *then* $(\phi \mid \chi) \cap (\psi \mid \chi) = \emptyset$

(Δ_3) *if* $\phi \neq \psi$, *then there exist co-final developments* ϱ *of* $\psi \mid \phi$, *and* ς *of* $\phi \mid \psi$

(Δ_4) *for* ϱ *and* ς *in* (Δ_3), $\chi \mid (\phi \cdot \varrho) = \chi \mid (\psi \cdot \varsigma)$

(J_1) *if* $\phi \, J \, \psi$, *then* $\phi \mid \psi$ *has precisely one member*

(J_2) *if* $\phi_1 \, J \, \phi_2$ *or* $\phi_1 = \phi_2$ *and* $\psi_1 \in \phi_1 \mid \chi$ *and* $\psi_2 \in \phi_2 \mid \chi$, *then* $\psi_1 \, J \, \psi_2$ *or* $\psi_1 = \psi_2$

Newman's Failure

(J_1) *If* $\xi J \eta$, $\xi \mid \eta$ *has precisely one member.*
(J_2) *If* $\eta_1 \in \xi_1 \mid \zeta$ *and* $\eta_2 \in \xi_2 \mid \zeta$, *and if* $\xi_1 J \xi_2$ *or* $\xi_1 = \xi_2$, *then* $\eta_1 J \eta_2$ *or* $\eta_1 = \eta_2$.

J represents non-nesting of redexes

Example (Schroer)

λ-calculus does not satisfy Newman's axioms
$\omega(\lambda y.\omega y) \to (\lambda y.\omega y)\lambda y.\omega y \to \underline{\omega(\lambda y.\omega y)} \to (\lambda y.\omega y)\lambda y.\omega y$
with $\omega = \lambda x.xx$

- by (J_2) derivates of ωy are (mutually) J-related.
- by (J_2) whole term and ωy-redex are (mutually) J-related.
- the ωy-redex is duplicated violating (J_1).

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Newman's Success / Redemption

Theorem

theorem does apply to Combinatory Logic (CL)

Newman's Success / Redemption

Theorem

theorem does apply to Combinatory Logic (CL)

Proof.

- $\phi \mid \psi :=$ residuals of ϕ after ψ
- ϕ *J* ψ if ϕ, ψ parallel to each other
  (formally: redexes at parallel positions)

□

Newman's Success / Redemption

Theorem

theorem does apply to Combinatory Logic (CL)

Proof.

- $\phi \mid \psi :=$ residuals of ϕ after ψ
- $\phi \, J \, \psi$ if ϕ, ψ parallel to each other

□

More Success

Theorem

$\langle \twoheadrightarrow, \emptyset, | \rangle$ *is a* 1*-ra, under assumptions of Newman's theorem*

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Theorem

$\langle \twoheadrightarrow\!\!\!\!\!\!+\!\!+, \emptyset, | \rangle$ *is a 1-ra, under assumptions of Newman's theorem*

Proof.

- *sets* Φ of J-related (co-initial) steps as steps of $\twoheadrightarrow\!\!\!\!\!\!+\!\!+$
- target of Φ is target of a(ny) development of Φ

□

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$\langle \twoheadrightarrow\!\!\!\!\!+\!\!\!\!+, \emptyset, | \rangle$ *is a 1-ra, under assumptions of Newman's theorem*

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□

Corollary

CL / OTRSs reductions have pushouts / least upperbounds
(theory of orthogonality applies)

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Theorem

$\langle \twoheadrightarrow\!\!\!\!+\!\!\!\!+, \emptyset, | \rangle$ *is a 1-ra, under assumptions of Newman's theorem*

Proof.

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- target of Φ is target of a(ny) development of Φ

$\square$

Corollary

CL / OTRSs reductions have pushouts / least upperbounds

seen via 1-ra(c)s above; now factored through Newman's axioms / result

1-algebra

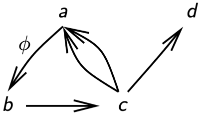
Definition

ARS $\longrightarrow$ is $\langle A, \Phi, \mathrm{src}, \mathrm{tgt}\rangle$

- A set of objects a, b, c, ...
- Φ set of steps ϕ, ψ, χ, ...
- src, tgt : $\Phi \longrightarrow A$
  source and target functions

$\phi : a \longrightarrow b$ denotes step ϕ with source a and target b

ARS is directed graph, e.g.



ISR 08 Obergurgl

1-algebra

Idea of 1-algebras

- algebra having rewrite system as carrier

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- 1-operations yielding steps

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1-algebra like algebra but then operating on steps instead of objects

1-algebra

Idea of 1-algebras

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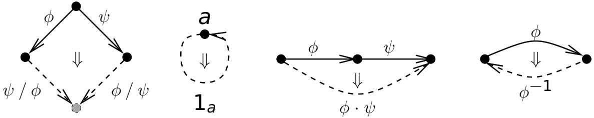
1-algebra operations of interest here: residuation, unit, composition, reverse



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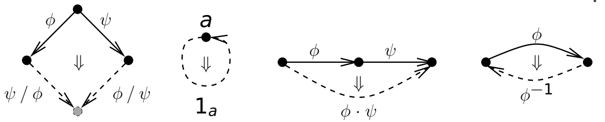

1-algebra laws of interest on 1-operations: those of 1-ra(c)s and also:

$$1 \cdot \varrho = \varrho \qquad 1^{-1} = 1$$

$$\varrho \cdot 1 = \varrho \qquad (\varrho \cdot \varsigma)^{-1} = \varsigma^{-1} \cdot \varrho^{-1}$$

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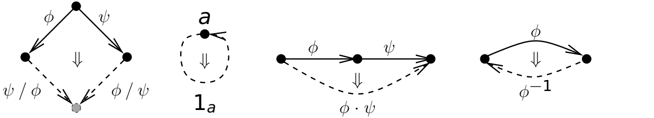

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algebra terminology reuse: speak of 1-monoid, 1-involution etc.
(category is a 1-monoid)

Freely constructing rewrite relations

freely constructing rewrite relations from rewrite relation $\rightarrow$

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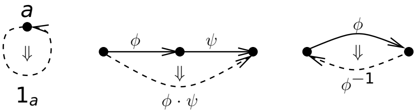
new relations constructed by closures
(least relation extending $\to$ having properties; universality)

Freely constructing rewrite systems

freely constructing rewrite systems from rewrite system →



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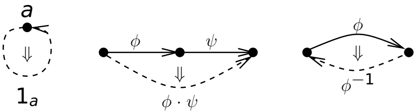
Freely constructing rewrite systems

freely constructing rewrite systems from rewrite system $\to$

$\langle \leftrightarrow, {}^{-1} \rangle$ is free 1-involutoid generated by $\to$



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Freely constructing rewrite systems

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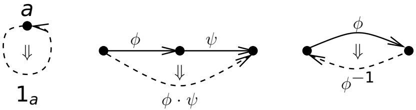
$\langle \twoheadrightarrow, 1, \cdot \rangle$ is free 1-monoid generated by $\to$
(reduction; path category over multidigraph)



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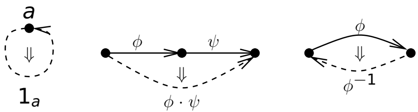
$\langle \leftrightarrow^*, 1, {}^{-1}, \cdot \rangle$ is free 1-involutive 1-monoid generated by $\rightarrow$ (conversion; dagger category)



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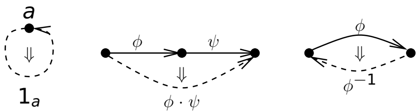
Freely constructing rewrite systems

freely constructing rewrite systems from rewrite system $\rightarrow$

new systems constructed by free generation of 1-algebras
(universality: map to such a 1-algebra factors uniquely through the free one)



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- WiP: lift $\mathbb{B} \hookrightarrow \mathbb{N} \hookrightarrow \mathbb{Z}$ for 1-ras
  (works for bits and braids)