Term Evaluation Systems with Refinements for Contextual Improvement by Critical Pair Analysis

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(based on the paper in FLOPS 2024)

Vefication Problem in FP

Functional program

```
[]       ++ ys ⟶ ys
(x:xs) ++ ys ⟶ x : (xs ++ ys)
```

Verify equality:

```
(xs ++ ys) ++ zs = xs ++ (ys ++ zs)
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- ▷ Inductive theorem?
- ▷ Real programming languages use **deterministic evaluation** mecahnisms
  - Call-by-value: OCaml
  - Call-by-name, call-by-need: Haskell
- ▷ HO functions, polymorphsim, e.g.

```
map f (map g xs) = map (f o g) xs
```

Vefication Problem in FP

Functional program

```
[]     ++ ys ⟶ ys
(x:xs) ++ ys ⟶ x : (xs ++ ys)
```

Verify **improvement**:

```
(xs ++ ys) ++ zs ⇛ xs ++ (ys ++ zs)
```

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```

Term Evaluation and Refinement System (TERS)

Signature Σ — $[\,]: 0,\ (:): 2,\ (+\!\!+): 2$

Values Val — $V ::= [\,] \mid V : V$

Evaluation contexts $Ectx$ — $E ::= \square \mid E +\!\!+ t \mid V +\!\!+ E \mid E : t \mid V : E$

Evaluation rules $\mathcal{E}$

$$[\,] +\!\!+ ys \to ys$$

$$(x : xs) +\!\!+ ys \to x : (xs +\!\!+ ys)$$

Refinement rule $\mathcal{R}$

$$(xs +\!\!+ ys) +\!\!+ zs \Rightarrow xs +\!\!+ (ys +\!\!+ zs)$$

Q. Is $\mathcal{R}$ contextual improvement?

Term Evaluation System (TERS)

Evaluation

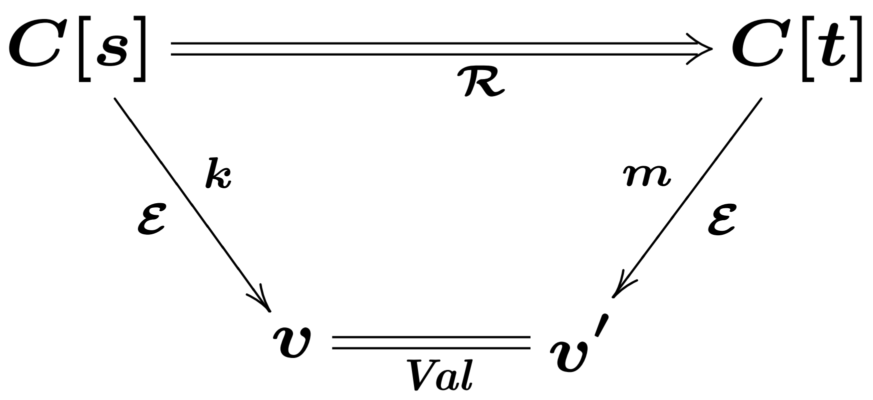
$$\frac{(l \longrightarrow r) \in \mathcal{E} \quad E \in Ectx}{E[l\theta] \longrightarrow_{\mathcal{E}} E[r\theta]}$$

Refinement

$$\frac{(l \Rightarrow r) \in \mathcal{R} \quad C \in Ctx}{C[l\theta] \Rightarrow_{\mathcal{R}} C[r\theta]}$$

Contextual Improvement

A set $\mathcal{R}$ of refinement rules is **contextual improvement** w.r.t. a set $\mathcal{E}$ evaluation rules if for all contexts C,

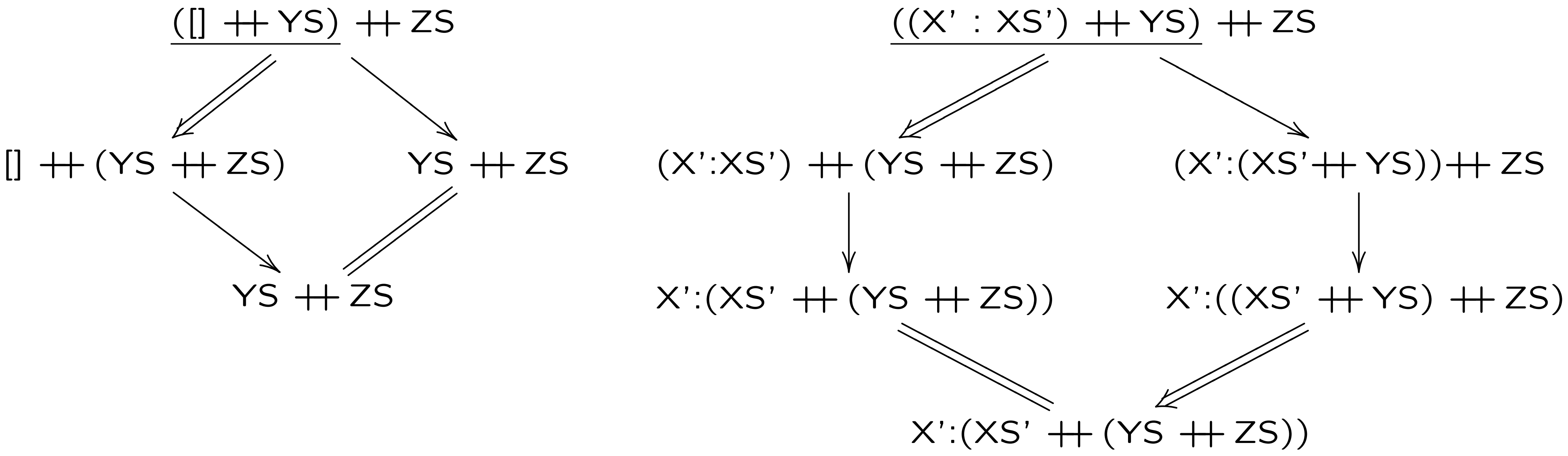


with $k \geq m$

- Difficult to prove because of universal quantification of contexts

Is $\mathcal{R}$ contextual improvement?

Yes, by checking critical pairs between $\mathcal{E}$ and $\mathcal{R}$



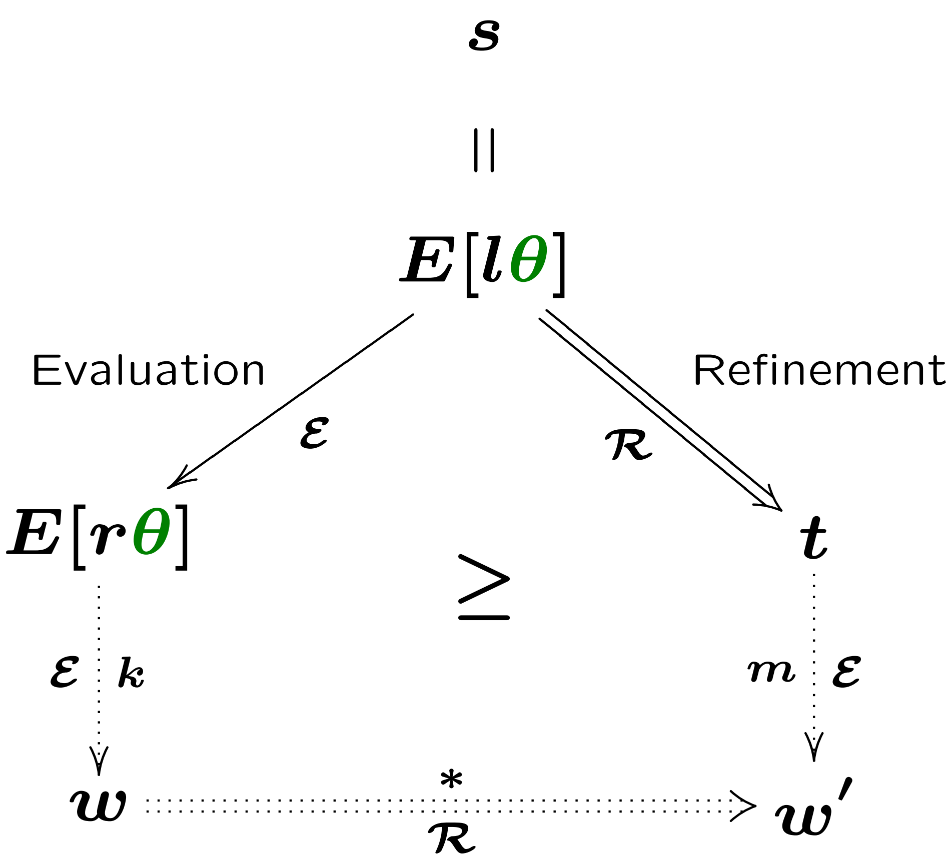
Main Theorem [FLOPS'24, Muroya&H.]

If

1. $\mathcal{E}$ is deterministic (ensured by evaluation contexts)
2. $\mathcal{R}$ is value-invariant (a refinement of value is a value)
3. TERS $(\mathcal{E}, \mathcal{R})$ is **locally coherent**

then $\mathcal{R}$ is **contextual improvement** wrt $\mathcal{E}$.

Local Coherence



with $\mathbf{1} + \boldsymbol{k} \geq \boldsymbol{m}$

- How to check?

Lemma [FLOPS'24, Muroya&H.]

A **well-behaved** TERS is locally coherent iff all its critical pairs are joinable.

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Well-behaved TERS $(\mathcal{E}, \mathcal{R})$

- Evaluation contexs are defined inductively.
- Refinement respects evaluation contexts: $\mathit{Ectx} \ni E \Rightarrow_{\mathcal{R}} E' \in \mathit{Ectx}$
- $\mathcal{R}$ linear (wrt non-value metavariable)
- $\mathcal{E}$ left-linear (wrt non-value metavariable)
- For every non-value metavariable M,
  if $l[M \mapsto \overline{x}.\square] \in \mathit{Ectx}$ then $r[M \mapsto \overline{x}.\square] \in \mathit{Ectx}$.

Highlights

- Developed this critical pair analysis method of **contextual improvement** for FO and SO TERS
- Developed tool ReCheck as an extension of SOL
- Checked various examples
  - Haskell: map/map, lazy lists, diverge, $\cdots$
  - Calculi: left-to-right call-by-value $\boldsymbol{\lambda}$-calculus, call-by-need $\boldsymbol{\lambda}$-calculus, computaional $\boldsymbol{\lambda}$-caluclus, effect handers

Example: Map/Map Fusion

```
map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = f x : map f xs

(.) :: (b -> c) -> (a -> b) -> (a -> c)
(.) f g x = f (g x)

{-# RULES
"map/map" forall f g xs. map f (map g xs) = map (f . g) xs #-}
```

Demo of ReCheck

Lazy program [Kikuchi, Sasano, Aoto, PPDP19]

```
(rep1)  replicate(z)      => []
(rep2)  replicate(s(N))   => s(z) : replicate(N)
(take1) take(z, XS)       => []
(take2) take(s(N), [])    => []
(take3) take(s(N), X:XS) => X : take(N,XS)
(ones)  ones => s(z) : ones
```

Refinement rule using the infinite list `ones`

```
(conj)  take(N, ones) => replicate(N)
```

- N elements from `ones` is refined to N copies of `s(z)`.
- Inductive theorem proving without SN
- ReCheck reports 2 CPs and 1 non-joinable
- Adding

```
take(N,s(z):ones) => replicate(N)
```

all joinable, then contextual improvement

Simulating Context-Sensitive Rewriting

Prop. Let $(\Sigma, \mathcal{R}, \mu)$ be a context-sensitive TRS [Lucus'92,...] and $(\Sigma, \mathcal{E}, Ectx, Val)$ be the correponding nondeterministic TES. We have:

$$t \hookrightarrow_{\mathcal{R},\mu} u \quad \Longleftrightarrow \quad t \rightarrow_{\mathcal{E}} u$$

Idea: replacement maps μ are simulated by evaluation contexts

Example

- ▷ Context-sensitive $\Rightarrow$ Term Evalation System
- ▷ $\mu(\text{if}) = \{1\}$ $\Leftrightarrow$ $E ::= \Box \mid \text{if}(E, t_1, t_2) \mid \cdots$
- ▷ $\mu(+) = \{1, 2\}$ $\Leftrightarrow$ $E ::= \Box \mid E + t \mid t + E \mid \cdots$ (non-determistic TES)
- ▷ $\mu(+) = \{1, 2\}$ $\Leftarrow$ $E ::= \Box \mid E + t \mid v + E \mid \cdots$ (determistic TES) with starategy

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▶ Based on this simulation, SOL 2025 (TERS evaluation, local coherence) has also the feature of checking CS confluence ▶ Prticipate CoCo 2025, CSR category

Future Work

- Alternative to (higher-order polymorphic) inductive theorem proving
- Rewriting properties on TERS: confluence, termination, etc.
- **Robust** Haskell's rewrite rule verifier
- Detailed comparison with FP verifier based on SMT solver
  Mochi, RCaml [Sato,Unno, Kobayashi 12,...],
  Timbuk [Haudebourg, Genet, Jenen 20]
  - These tools are good on problem involving interger constraints, but weak on ones involving algebraic datatypes
  - E.g. These could not verify `copy(copy(N) = N)` under `copy(0) = 0; copy(s(N)) = s(copy(N))`