Confluence of 001- and 101-infinitary λ-calculi by linear approximation

Rémy Cerda and Lionel Vaux Auclair

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Infinitary λ-calculi

Head reduction reduces head redexes

$$\lambda\vec{x}.(\lambda y.P)QM_1 \dots M_n$$

unless we see a head normal form (HNF)

$$\lambda\vec{x}.yM_1 \dots M_n.$$

The full evaluation of M is given by its **Böhm tree**

$$\mathrm{BT}(M) := \begin{cases} \lambda\vec{x}.y\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n) & \\ \qquad\qquad\qquad \text{if } M \longrightarrow^{*}_{\beta} \text{HNF}, \\ \bot \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Strict and lazy evaluation

Head reduction reduces head redexes

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unless we see a head normal form (HNF)

$$\lambda\vec{x}.yM_1 \dots M_n.$$

The full evaluation of M is given by its **Böhm tree**

$$\mathrm{BT}(M) := \begin{cases} \lambda\vec{x}.y\mathrm{BT}(M_1)\dots\mathrm{BT}(M_n) & \text{if } M \longrightarrow_\beta^* \text{HNF}, \\ \bot & \text{otherwise.} \end{cases}$$

Weak head reduction reduces weak head redexes

$$(\lambda y.P)QM_1 \dots M_n$$

unless we see a weak head normal form (WHNF)

$$\lambda x.M' \quad \text{or} \quad yM_1 \dots M_n.$$

The full evaluation of M is given by its **Lévy-Longo tree**

$$\mathrm{LLT}(M) := \begin{cases} \lambda x.\mathrm{LLT}(M') & \text{if } (\dots), \\ y\mathrm{LLT}(M_1)\dots\mathrm{LLT}(M_n) & \text{if } (\dots), \\ \bot & \text{otherwise.} \end{cases}$$

Consider **001-infinitary λ⊥-terms**:

$$\frac{}{x \in \Lambda_\bot^{001}} \qquad \frac{P \in \Lambda_\bot^{001}}{\lambda x.P \in \Lambda_\bot^{001}} \qquad \frac{P \in \Lambda_\bot^{001} \qquad \underline{Q \in \Lambda_\bot^{001}}}{PQ \in \Lambda_\bot^{001}} \qquad \frac{}{\bot \in \Lambda_\bot^{001}}$$

together with **001-infinitary β⊥-reduction**:

$$\longrightarrow_{\beta\bot} := \longrightarrow_\beta + \{M \longrightarrow \bot \mid M \text{ has no HNF}\} + \text{lifting to contexts}$$

$$\frac{M \longrightarrow_{\beta\bot}^* N}{M \longrightarrow_{\beta\bot}^{001} N} \qquad \frac{M \longrightarrow_{\beta\bot}^* \lambda x.P \qquad P \longrightarrow_{\beta\bot}^{001} P'}{M \longrightarrow_{\beta\bot}^{001} \lambda x.P'} \qquad \frac{M \longrightarrow_{\beta\bot}^* PQ \qquad P \longrightarrow_{\beta\bot}^{001} P' \qquad \underline{Q \longrightarrow_{\beta\bot}^{001} Q'}}{M \longrightarrow_{\beta\bot}^{001} P'Q'}$$

Theorem [KKSdV'97]

$\longrightarrow_{\beta\bot}^{\infty}$ is confluent, and $BT(M)$ is the unique infinitary β⊥-nf of M.

Consider **101-infinitary λ⊥-terms**:

$$\frac{}{x \in \Lambda_\perp^{101}} \qquad \frac{P \in \Lambda_\perp^{101}}{\lambda x.P \in \Lambda_\perp^{101}} \qquad \frac{P \in \Lambda_\perp^{101} \qquad Q \in \Lambda_\perp^{101}}{PQ \in \Lambda_\perp^{101}} \qquad \frac{}{\perp \in \Lambda_\perp^{101}}$$

together with **101-infinitary β⊥-reduction**:

$$\longrightarrow_{\beta\perp} := \longrightarrow_\beta + \{M \longrightarrow \perp \mid M \text{ has no WHNF}\} + \text{lifting to contexts}$$

$$\frac{M \longrightarrow_{\beta\perp}^{*} N}{M \longrightarrow_{\beta\perp}^{101} N} \qquad \frac{M \longrightarrow_{\beta\perp}^{*} \lambda x.P \qquad P \longrightarrow_{\beta\perp}^{101} P'}{M \longrightarrow_{\beta\perp}^{101} \lambda x.P'} \qquad \frac{M \longrightarrow_{\beta\perp}^{*} PQ \qquad P \longrightarrow_{\beta\perp}^{101} P' \qquad Q \longrightarrow_{\beta\perp}^{101} Q'}{M \longrightarrow_{\beta\perp}^{101} P'Q'}$$

Theorem [KKSdV'97]

$\longrightarrow_{\beta\perp}^{\infty}$ is confluent, and LLT(M) is the unique infinitary β⊥-nf of M.

Linear approximation

linearity!

Linear approximation provides a nice refinement of continuous approximation by taking λ-terms to a sum of "multilinear λ-terms", aka **resource terms**:

$$s, t, \ldots \quad := \quad x \quad | \quad \lambda x.s \quad | \quad s[t_1, \ldots, t_n].$$

$$\phi(x) := x$$
$$\phi(\lambda x.P) := \lambda x.\phi(M)$$
$$\phi(PQ) := \phi(P)[\phi(Q)]$$
$$\phi(P\bot) := \phi(P)[]$$

- **Multilinear substitution:**

$$s\langle[t_1,\dots,t_n]/x\rangle := \begin{cases} \sum_{\sigma\in\mathfrak{S}(n)} s[t_{\sigma(1)}/x_1,\dots,t_{\sigma(n)}/x_n] & \text{if } \deg_x(s) = n \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

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- **Resource reduction:** $(\lambda x.s)\bar{t} \longrightarrow_{\mathrm{r}} s\langle\bar{t}/x\rangle$ + lifting to contexts and fin. sums.
This relation $\longrightarrow_{\mathrm{r}}$ is strongly confluent and strongly normalising.

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- **Resource reduction:** $(\lambda x.s)\bar{t} \longrightarrow_r s\langle\bar{t}/x\rangle$ + lifting to contexts and fin. sums.
  This relation $\longrightarrow_r$ is strongly confluent and strongly normalising.

The **Taylor expansion** of M is the set $\mathcal{T}(M) := \{s \in \Lambda_r \mid s \sqsubseteq_{\mathcal{T}} M\}$, with:

$$\frac{}{x \sqsubseteq_{\mathcal{T}} x} \qquad \frac{s \sqsubseteq_{\mathcal{T}} M}{\lambda x.s \sqsubseteq_{\mathcal{T}} \lambda x.M} \qquad \frac{s \sqsubseteq_{\mathcal{T}} M \quad t_1 \sqsubseteq_{\mathcal{T}} N \quad \dots \quad t_n \sqsubseteq_{\mathcal{T}} N}{(s)[t_1, \dots, t_n] \sqsubseteq_{\mathcal{T}} (M)N}$$

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- **Lifting to sets:** $\bigcup_i \{s_i\} \longrightarrow\!\!\!\!\!\rightarrow_{\mathrm{r}} \bigcup_i |\mathbf{t}_i|$ whenever $\forall i,\ s_i \longrightarrow_{\mathrm{r}}^* \mathbf{t}_i$.

The big theorem of "the linear approximation of the λ-calculus":

Commutation theorem [ER'06]

$\mathrm{nf}_{\mathrm{r}}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$.

can be improved thanks to the introduction of the infinitary λ-calculus:

Simulation theorem [CV'23]

If $M \longrightarrow^{001}_{\beta\bot} N$ then $\mathcal{T}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}(N)$.

The same results, lazily

If $M \longrightarrow^{101}_{\beta\bot} N$ then $\mathcal{T}_{\ell}(M) \twoheadrightarrow_{\mathrm{r}} \mathcal{T}_{\ell}(N)$.
Corollary, $\mathrm{nf}_{\mathrm{r}}(\mathcal{T}_{\ell}(M)) = \mathcal{T}_{\ell}(\mathrm{LLT}(M))$.

But also: conversely, linear approximation entails confluence of the 001- and 101-infinitary λ-calculi.

How linearity acts

An example (using a fixed-point combinator Y and $\mathsf{K} := \lambda xy.x$):

$$\mathsf{YK} \longrightarrow_{\beta}^{a01} \mathsf{K}^{\omega} := K(K(K(\dots))) \qquad \mathsf{YK} \longrightarrow_{\beta}^{*} (\lambda xy.xx)(\lambda xy.xx)$$

If $a = 0$ this is a critical pair. Confluence is restored by $\bot$-reductions:

$$\mathsf{K}^{\omega} \longrightarrow_{\mathsf{h}} \lambda y.\mathsf{K}^{\omega} \longrightarrow_{\bot} \bot \qquad (\lambda xy.xx)(\lambda xy.xx) \longrightarrow_{\mathsf{h}} \lambda y.\text{itself} \longrightarrow_{\bot} \bot.$$

This is simulated by $\mathcal{T}(\mathsf{K}^{\omega}) \longrightarrow_{\mathsf{r}} \emptyset$, indeed:

$$\mathcal{T}(\mathsf{K}^{\omega}) \overset{\text{ind.}}{=} \{\mathsf{K}[t_1, \dots, t_n] \mid n \in \mathbf{N},\ t_1, \dots, t_n \in \mathcal{T}(\mathsf{K}^{\omega})\}$$

the base case being $\mathsf{K}[] \longrightarrow \mathbf{0}$, hence every term in $\mathcal{T}(\mathsf{K}^{\omega})$ vanishes by linearity.

If $a = 1$ everything's fine again. $\mathsf{O} := \lambda y_0.\lambda y_1.\lambda y_2.\dots$ is a common reduct.

Theorem (uniqueness of normal forms)

For all $M \in \Lambda_{\perp}^{001}$, $\mathrm{BT}(M)$ is the unique normal form for $\longrightarrow_{\beta\perp}$ reachable through $\longrightarrow_{\beta\perp}^{001}$ from M.

Proof. Suppose there is another such normal form, denote it by N. Then:

$$\mathcal{T}(N) = \mathcal{T}(\mathrm{BT}(N)) = \mathrm{nf}_{\mathrm{r}}(\mathcal{T}(N)) = \mathrm{nf}_{\mathrm{r}}(\mathcal{T}(M)) = \mathcal{T}(\mathrm{BT}(M))$$

and finally $N = \mathrm{BT}(M)$.

Corollary (confluence). $\longrightarrow_{\beta\perp}^{001}$ is confluent on $\Lambda_{\perp}^{001}$.

The same result, lazily. $\longrightarrow_{\beta\perp}^{101}$ is confluent on $\Lambda_{\perp}^{101}$.

Beyond 001 and 101?

A **meaningless set** is a set $\mathcal{U}$ of λ-terms s.t. [KOV'99, SV'11]

- all the very bad terms are in $\mathcal{U}$,
- $\mathcal{U}$ is closed under (...).

$\longrightarrow_{\beta\bot\mathcal{U}}$ is $\longrightarrow_{\beta}$ + $\dfrac{M \in \mathcal{U}}{M \longrightarrow_{\beta\bot\mathcal{U}} \bot}$ + lifting to contexts.

$\longrightarrow^{\infty}_{\beta\bot\mathcal{U}}$ is its (111-)infinitary closure.

Theorem

$\longrightarrow^{\infty}_{\beta\bot\mathcal{U}}$ is confluent.

Hence each M has a unique $\beta\bot_{\mathcal{U}}$-nf, denoted by $\mathrm{T}_{\mathcal{U}}(M)$.

This induces a **normal form model**.

No Taylor expansion outside the strict and lazy cases

Unsurprising examples:

$$\overline{\mathcal{HN}} := \{M \in \Lambda^{\infty} \mid M \text{ has no } \textsc{hnf}\} \qquad \overline{\mathcal{WN}} := \{M \in \Lambda^{\infty} \mid M \text{ has no } \textsc{whnf}\}$$

$$\mathrm{T}_{\overline{\mathcal{HN}}} = \mathrm{BT} \qquad \mathrm{T}_{\overline{\mathcal{WN}}} = \mathrm{LLT}$$

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$$\mathrm{T}_{\overline{\mathcal{HN}}} = \mathrm{BT} \qquad \mathrm{T}_{\overline{\mathcal{WN}}} = \mathrm{LLT}$$

One more corollary. $\mathrm{LLT} : \Lambda^\infty \to \Lambda^\infty$ (and similarly BT) is Scott-continuous.

Proof.

For all directed D, observe that $\mathcal{T}(\bigsqcup D) = \bigcup \mathcal{T}(D)$.

Conclude using this and Commutation. □

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Unsurprising examples:

$$\overline{\mathcal{HN}} := \{M \in \Lambda^\infty \mid M \text{ has no } \textsc{hnf}\} \qquad \overline{\mathcal{WN}} := \{M \in \Lambda^\infty \mid M \text{ has no } \textsc{whnf}\}$$

$$\mathrm{T}_{\overline{\mathcal{HN}}} = \mathrm{BT} \qquad \mathrm{T}_{\overline{\mathcal{WN}}} = \mathrm{LLT}$$

One more corollary. LLT $: \Lambda^\infty \to \Lambda^\infty$ (and similarly BT) is Scott-continuous.

Proof.
For all directed D, observe that $\mathcal{T}(\bigsqcup D) = \bigcup \mathcal{T}(D)$.
Conclude using this and Commutation. □

Theorem. $\mathrm{T}_{\mathcal{U}}$ is Scott continuous only when $\mathcal{U}$ is $\overline{\mathcal{HN}}$ or $\overline{\mathcal{WN}}$. [SV'05]

Hence **there is no (reasonable) Taylor expansion for more than BTs and LLTs!**

Why I am presenting this

- Linearity makes confluence very easy...
  but linear approximation is very strong/constrained
  so maybe it is not such a great general technique for proving confluence 🙃
- However linear approximation is much more general than it looks like 😊:
  available for
    - η
    - probabilistic, quantum, algebraic λ-calculi
    - $\Lambda\mu$-calculus
    - process calculi (in particular the very general one by [DM'24])

  and the connection between infinitary rewriting and approximation techniques is under-exploited

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